

# Measure-valued equations for Kolmogorov operators with unbounded coefficients

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## Abstract

Given a real and separable Hilbert space  $H$  we consider the measure-valued equation

$$\int_H \varphi(x) \mu_t(dx) - \int_H \varphi(x) \mu(dx) = \int_0^t \left( \int_H K_0 \varphi(x) \mu_s(dx) \right) ds,$$

where  $K_0$  is the Kolmogorov differential operator

$$K_0 \varphi(x) = \frac{1}{2} \text{Trace}[BB^* D^2 \varphi(x)] + \langle x, A^* D \varphi(x) \rangle + \langle D \varphi(x), F(x) \rangle,$$

$x \in H$ ,  $\varphi : H \rightarrow \mathbb{R}$  is a suitable smooth function,  $A : D(A) \subset H \rightarrow H$  is linear,  $F : H \rightarrow H$  is a globally Lipschitz function and  $B : H \rightarrow H$  is linear and continuous. In order to prove existence and uniqueness of a solution for the above equation, we show that  $K_0$  is a core, in a suitable way, of the infinitesimal generator associated to the solution of a certain stochastic differential equation in  $H$ .

We also extend the above results to a reaction-diffusion operator with polynomial nonlinearities.

## 1 Introduction

Let  $H$  be a separable real Hilbert space (with norm  $|\cdot|$  and inner product  $\langle \cdot, \cdot \rangle$ ), and let  $\mathcal{B}(H)$  be its Borel  $\sigma$ -algebra.  $\mathcal{L}(H)$  denotes the usual Banach space of all linear and continuous operators in  $H$ , endowed with the

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supremum norm  $\|\cdot\|_{\mathcal{L}(H)}$ . We consider the stochastic differential equation in  $H$

$$\begin{cases} dX(t) = (AX(t) + F(X(t)))dt + BdW(t), & t \geq 0 \\ X(0) = x \in H, \end{cases} \quad (1)$$

where

**Hypothesis 1.1.** (i)  $A: D(A) \subset H \rightarrow H$  is the infinitesimal generator of a strongly continuous semigroup  $e^{tA}$  of type  $\mathcal{G}(M, \omega)$ , i.e. there exist  $M \geq 0$  and  $\omega \in \mathbb{R}$  such that  $\|e^{tA}\|_{\mathcal{L}(H)} \leq Me^{\omega t}$ ,  $t \geq 0$ ;

(ii)  $B \in \mathcal{L}(H)$  and for any  $t > 0$  the linear operator  $Q_t$ , defined by

$$Q_t x = \int_0^t e^{sA} B B^* e^{sA^*} x ds, \quad x \in H, \quad t \geq 0$$

has finite trace;

(iii)  $F: H \rightarrow H$  is a Lipschitz continuous map. We set

$$\kappa = \sup_{\substack{x, y \in H \\ x \neq y}} \frac{|F(x) - F(y)|}{|x - y|};$$

(iv)  $(W(t))_{t \geq 0}$  is a cylindrical Wiener process, defined on a stochastic basis  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$  and with values in  $H$ .

It is well known that under hypothesis (1.1) problem (1) has a unique mild solution  $(X(t, x))_{t \geq 0, x \in H}$  (see, for instance, [7]), that is for any  $x \in H$  the process  $(X(t, x))_{t \geq 0}$  is adapted to the filtration  $(\mathcal{F}_t)_{t \geq 0}$ , it is continuous in mean square and it fulfils the integral equation

$$X(t, x) = e^{tA}x + \int_0^t e^{(t-s)A} B dW(s) + \int_0^t e^{(t-s)A} F(X(s, x)) ds \quad (2)$$

for any  $t \geq 0$ . Moreover, a straightforward computation shows that for any  $T > 0$  there exists  $c > 0$  such that

$$\sup_{t \in [0, T]} |X(t, x) - X(t, y)| \leq c|x - y|, \quad \forall x, y \in H, \quad (3)$$

and

$$\sup_{t \in [0, T]} \mathbb{E}[|X(t, x)|] \leq c(1 + |x|), \quad x \in H, \quad (4)$$

where the expectation is taken with respect to  $\mathbb{P}$ . Now denote by  $C_b(H)$  the Banach space of all uniformly continuous and bounded functions  $\varphi$ :

$H \rightarrow \mathbb{R}$ , endowed with the supremum norm  $\|\varphi\|_0 = \sup_{x \in H} |\varphi(x)|$ ,  $\varphi \in C_b(H)$ . Moreover, for any  $k > 0$ , let  $C_{b,k}(H)$  be the space of all functions  $\varphi : H \rightarrow \mathbb{R}$  such that the function  $H \rightarrow \mathbb{R}$ ,  $x \mapsto (1 + |x|^k)^{-1} \varphi(x)$  belongs to  $C_b(H)$ . The space  $C_{b,k}(H)$  is a Banach space, endowed with the norm  $\|\varphi\|_{0,k} = \|(1 + |\cdot|^k)^{-1} \varphi\|_0$ . In the following, we shall denote by  $(C_{b,k}(H))^*$  the topological dual space of  $C_{b,k}(H)$ . As we shall see in Proposition 2.2, estimates (3), (4) allow us to define the transition operator associated to equation (2) in the space  $C_{b,1}(H)$ , by the formula

$$P_t \varphi(x) = \mathbb{E}[\varphi(X(t, x))], \quad \varphi \in C_{b,1}(H), \quad t \geq 0, \quad x \in H. \quad (5)$$

The family of operators  $(P_t)_{t \geq 0}$  maps  $C_{b,1}(H)$  into  $C_{b,1}(H)$  and enjoys the semigroup property, but it is not a strongly continuous semigroup (cf Proposition 2.2). However, we can define the infinitesimal generator of  $(P_t)_{t \geq 0}$  in  $C_{b,1}(H)$  in the following way

$$\left\{ \begin{aligned} D(K) &= \left\{ \varphi \in C_{b,1}(H) : \exists g \in C_{b,1}(H), \lim_{t \rightarrow 0^+} \frac{P_t \varphi(x) - \varphi(x)}{t} = \right. \\ &\quad \left. = g(x), \quad x \in H, \quad \sup_{t \in (0,1)} \left\| \frac{P_t \varphi - \varphi}{t} \right\|_{0,1} < \infty \right\} \\ K\varphi(x) &= \lim_{t \rightarrow 0^+} \frac{P_t \varphi(x) - \varphi(x)}{t}, \quad \varphi \in D(K), \quad x \in H. \end{aligned} \right. \quad (6)$$

Let  $\mathcal{M}(H)$  be the space of all the Borel finite measures on  $H$  and for any  $k > 0$  let  $\mathcal{M}_k(H)$  be the set of all  $\mu \in \mathcal{M}(H)$  such that  $\int_H |x|^k |\mu|(dx) < \infty$ , where  $|\mu|$  is the total variation of  $\mu$ . The first result of the paper is the following

**Theorem 1.2.** *Let  $(P_t)_{t \geq 0}$  be the semigroup defined by (5) and let  $(K, D(K))$  be its infinitesimal generator in  $C_{b,1}(H)$ , defined by (6). Then, the formula*

$$\langle \varphi, P_t^* F \rangle_{\mathcal{L}(C_{b,1}(H), (C_{b,1}(H))^*)} = \langle P_t \varphi, F \rangle_{\mathcal{L}(C_{b,1}(H), (C_{b,1}(H))^*)}$$

*defines a semigroup  $(P_t^*)_{t \geq 0}$  of linear and continuous operators on  $(C_{b,1}(H))^*$  that maps  $\mathcal{M}_1(H)$  into  $\mathcal{M}_1(H)$ . Moreover, for any  $\mu \in \mathcal{M}_1(H)$  there exists a unique family of measures  $\{\mu_t\}_{t \geq 0} \subset \mathcal{M}_1(H)$  such that*

$$\int_0^T \left( \int_H |x| |\mu_t|(dx) \right) dt < \infty, \quad \forall T > 0 \quad (7)$$

*and*

$$\int_H \varphi(x) \mu_t(dx) - \int_H \varphi(x) \mu(dx) = \int_0^t \left( \int_H K\varphi(x) \mu_s(dx) \right) ds \quad (8)$$

*for any  $t \geq 0$ ,  $\varphi \in D(K)$ . Finally, the solution of (8) is given by  $\{P_t^* \mu\}_{t \geq 0}$ .*

A natural question is to study the above problem with the *Kolmogorov* differential operator

$$K_0\varphi(x) = \frac{1}{2}\text{Tr}[BB^*D^2\varphi(x)] + \langle x, A^*D\varphi(x) \rangle + \langle D\varphi(x), F(x) \rangle, \quad x \in H. \quad (9)$$

We stress the fact that the operator  $K$  is defined in an abstract way, whereas  $K_0$  is a *concret* differential operator.

In order to study problem (8) with  $K_0$  replacing  $K$ , we shall develop the notion of  $\pi$ -convergence in the spaces  $C_{b,k}(H)$  and the related notion of  $\pi$ -core. We recall that the  $\pi$ -convergence has been introduced in [10], in order to study a class of semigroups that are not strongly continuous. This notion is one of the key tools we use to prove our results.

Now let  $\mathcal{E}_A(H)$  be the linear span of the real and imaginary part of the functions

$$H \rightarrow \mathbb{C}, \quad x \mapsto e^{i\langle x, h \rangle}, \quad h \in D(A^*),$$

where  $D(A^*)$  is the domain of the adjoint operator of  $A$ . We have the following

**Theorem 1.3.** *Under Hypothesis (1.1), the operator  $(K, D(K))$  is an extension of  $K_0$ , and for any  $\varphi \in \mathcal{E}_A(H)$  we have  $\varphi \in D(K)$  and  $K\varphi = K_0\varphi$ . Finally,  $\mathcal{E}_A(H)$  is a  $\pi$ -core for  $(K, D(K))$ .*

As consequence we have the third main result

**Theorem 1.4.** *For any  $\mu \in \mathcal{M}_1(H)$  there exists an unique family of measures  $\{\mu_t\}_{t \geq 0} \subset \mathcal{M}_1(H)$  fulfilling (7) and the measure equation*

$$\int_H \varphi(x) \mu_t(dx) - \int_H \varphi(x) \mu(dx) = \int_0^t \left( \int_H K_0\varphi(x) \mu_s(dx) \right) ds, \quad (10)$$

*$t \geq 0$ ,  $\varphi \in \mathcal{E}_A(H)$ , and the solution is given by  $\{P_t^*\mu\}_{t \geq 0}$ .*

In [9] a similar problem when  $F : H \rightarrow H$  is Lipschitz continuous and bounded has been investigated. Due to the fact that the nonlinearity is bounded, all the result are stated in the space  $C_b(H)$ . In our paper we deal with unbounded nonlinearities and we need to develop a notion of semigroup and associated infinitesimal generator in the weighted space  $C_{b,1}(H)$ .

In section 6 we shall extend the techniques and the results of the preceding sections to a reaction-diffusion operator with polinomial nonlinearities.

The motivation of this work is to have a better understanding on the relationships between the stochastic differential equation (1) and the Kolmogorov differential operator  $K_0$ . In this direction, the characterization done by Theorems 1.3, 6.3 seems to be new.

In other papers, the problem of extending a differential operator like (9) to the infinitesimal generator of a diffusion semigroup is studied in the weighted spaces  $L^p(H, \nu)$ ,  $p \geq 1$  where  $\nu$  is an invariant measure for the semigroup (see, for instance, [8] and references therein). Indeed, if  $\mu$  is an invariant measure for the semigroup (5), then the semigroup (5) can be extended to a strongly continuous contraction semigroup in  $L^p(H, \nu)$  whose generator is, say,  $(K_p, D(K_p))$ . It worth to notice that as consequence of Theorem 1.3, the set  $\mathcal{E}_A(H)$  is a core (with respect to the norm of  $L^p(H, \nu)$ ) for  $(K_p, D(K_p))$ .

Kolmogorov equations for measures in finite dimension have been the object of several papers. We recall that in the papers [2] have been stated sufficient conditions in order to ensure existence of a weak solution for partial differential operators of the form

$$H\varphi(t, x) = a^{ij}(t, x)\partial_{x_i}\partial_{x_j}\varphi(x) + b^i(t, x)\partial_{x_i}\varphi(x), \quad (t, x) \in (0, 1) \times \mathbb{R}^d$$

where  $\varphi \in C_0^\infty(\mathbb{R}^d)$  and  $a^{ij}, b^i: (0, 1) \times \mathbb{R}^d \rightarrow \mathbb{R}$ ,  $1 \leq i, j \leq d$  are suitable locally integrable functions. With similar techniques, in [1] the problem is studied for parabolic differential operators of the form  $Lu(t, x) = u_t(t, x) + Hu(t, x)$ ,  $u \in C_0^\infty((0, 1) \times \mathbb{R}^d)$ . The infinite dimensional framework has been investigated in [3], where it is considered an equation for measures of the form

$$\int_X L_{A,B}\psi(x)\mu(dx) = 0, \quad \forall \psi \in \mathcal{K},$$

where  $X$  is a locally convex space,  $\mathcal{K}$  is a suitable set of cylindrical functions and  $L_{A,B}$  is formally given by

$$L_{A,B}\psi(x) = \sum_{i,j=1}^{\infty} A_{i,j}\partial_{e_i}\partial_{e_j}\psi(x) + \sum_{i=1}^{\infty} B_i\partial_{e_i}\psi(x),$$

with  $\mu$ -measurable functions  $A_{i,j}$ ,  $B_i$  and vectors  $e_i \in X$ . Under some integrability assumptions on  $A_{i,j}$ ,  $B_i$ , the authors prove existence of a measure  $\mu$ , possibly infinite, satisfying the above equation.

We stress that in our paper we prove *uniqueness* results. In this direction, the results of Theorems 1.4, 6.4 are, at our knowledge, new.

The paper is organized as follows: in the next section we introduce the notions of  $\pi$ -convergence and we prove some results about the transition semigroup (5) in the space  $C_{b,1}(H)$ . Sections 3, 4, 5 concern proofs of Theorems 1.2, 1.3, 1.4, respectively. Section 6 is devoted to extend the results to a reaction-diffusion operator.

## 2 Notations and preliminary results

If  $E$  is a Banach space, we denote by  $C_b(H; E)$  the Banach space of all uniformly continuous and bounded functions  $f : H \rightarrow E$ , endowed the usual supremum norm  $\|\cdot\|_{C_b(H; E)}$ .  $C_b^1(H)$  denotes the space of all the functions  $f \in C_b(H)$  which are Fréchet differentiable with uniformly continuous and bounded differential  $DF \in C_b(H; \mathcal{L}(H; E))$ .

We deal with semigroups of operators which are not, in general, strongly continuous. For this reason, we introduce the notion of  $\pi$ -convergence in the space  $C_b(H)$  (see [9], [10]).

**Definition 2.1.** A sequence  $(\varphi_n)_{n \in \mathbb{N}} \subset C_b(H)$  is said to be  $\pi$ -convergent to a function  $\varphi \in C_b(H)$  if for any  $x \in H$  we have

$$\lim_{n \rightarrow \infty} \varphi_n(x) = \varphi(x)$$

and

$$\sup_{n \in \mathbb{N}} \|\varphi_n\|_0 < \infty.$$

Similarly, the  $m$ -indexed sequence  $(\varphi_{n_1, \dots, n_m})_{n_1 \in \mathbb{N}, \dots, n_m \in \mathbb{N}} \subset C_b(H)$  is said to be  $\pi$ -convergent to  $\varphi \in C_b(H)$  if for any  $i \in \{2, \dots, m\}$  there exists an  $i-1$ -indexed sequence  $(\varphi_{n_1, \dots, n_{i-1}})_{n_1 \in \mathbb{N}, \dots, n_{i-1} \in \mathbb{N}} \subset C_b(H)$  such that

$$\lim_{n_i \rightarrow \infty} \varphi_{n_1, \dots, n_i} \stackrel{\pi}{=} \varphi_{n_1, \dots, n_{i-1}}$$

and

$$\lim_{n_1 \rightarrow \infty} \varphi_{n_1} \stackrel{\pi}{=} \varphi.$$

We shall write

$$\lim_{n_1 \rightarrow \infty} \cdots \lim_{n_m \rightarrow \infty} \varphi_{n_1, \dots, n_m} \stackrel{\pi}{=} \varphi$$

or  $\varphi_n \xrightarrow{\pi} \varphi$  as  $n \rightarrow \infty$ , when the sequence has one index.

Note that since the convergence is pointwise we can not take a diagonal sequence. However, in order to avoid eavy notations, we shall often assume that the sequence has one index.

As easily seen the  $\pi$ -convergence implies the convergence in  $L^p(H; \mu)$ , for any  $\mu \in \mathcal{M}(H)$ ,  $p \in [1, \infty)$ .

Let  $k > 0$ . We shall often use the fact that if for a sequence  $(\varphi_n)_{n \in \mathbb{N}} \subset C_{b,k}(H)$  we have that  $(1 + |x|^k)^{-1} \varphi_n \xrightarrow{\pi} \varphi \in C_{b,k}(H)$  as  $n \rightarrow \infty$ , then the sequence converges to  $\varphi$  in  $L^p(H; \mu)$ , for any  $\mu \in \mathcal{M}_k(H)$ ,  $p \in [1, \infty)$ . This argument may be viewed as an extension of the  $\pi$ -convergence to the spaces  $C_{b,k}(H)$ .

In Theorem 1.3 we claim that  $\mathcal{E}_A(H)$  is a  $\pi$ -core for  $(K, D(K))$ . This means that if  $\varphi \in \mathcal{E}_A(H)$  we have  $\varphi \in D(K)$  and  $K\varphi = K_0\varphi$ . In addition, if  $\varphi \in D(K)$ , there exist  $m \in \mathbb{N}$  and an  $m$ -indexed sequence  $(\varphi_{n_1, \dots, n_m})_{n_1 \in \mathbb{N}, \dots, n_m \in \mathbb{N}} \subset \mathcal{E}_A(H)$  such that

$$\lim_{n_1 \rightarrow \infty} \cdots \lim_{n_m \rightarrow \infty} \frac{\varphi_{n_1, \dots, n_m}}{1 + |\cdot|} \stackrel{\pi}{=} \frac{\varphi}{1 + |\cdot|}, \quad \lim_{n_1 \rightarrow \infty} \cdots \lim_{n_m \rightarrow \infty} \frac{K_0 \varphi_{n_1, \dots, n_m}}{1 + |\cdot|} \stackrel{\pi}{=} \frac{K\varphi}{1 + |\cdot|}.$$

The construction of such a sequence is detailed in section 4.

## 2.1 The transition semigroup in $C_{b,1}(H)$

This section is devoted to study the semigroup  $(P_t)_{t \geq 0}$  in the space  $C_{b,1}(H)$ .

**Proposition 2.2.** *Formula (5) defines a semigroup of operators  $(P_t)_{t \geq 0}$  in  $C_{b,1}(H)$ , and there exist a family of probability measures  $\{\pi_t(x, \cdot), t \geq 0, x \in H\} \subset \mathcal{M}_1(H)$  and two constants  $c_0 > 0, \omega_0 \in \mathbb{R}$  such that*

- (i)  $P_t \in \mathcal{L}(C_{b,1}(H))$  and  $\|P_t\|_{\mathcal{L}(C_{b,1}(H))} \leq c_0 e^{\omega_0 t}$ ;
- (ii)  $P_t \varphi(x) = \int_H \varphi(y) \pi_t(x, dy)$ , for any  $t \geq 0, \varphi \in C_{b,1}(H), x \in H$ ;
- (iii) for any  $\varphi \in C_{b,1}(H), x \in H$ , the function  $\mathbb{R}^+ \rightarrow \mathbb{R}, t \mapsto P_t \varphi(x)$  is continuous.
- (iv)  $P_t P_s = P_{t+s}$ , for any  $t, s \geq 0$  and  $P_0 = I$ ;
- (v) for any  $\varphi \in C_{b,1}(H)$  and any sequence  $(\varphi_n)_{n \in \mathbb{N}} \subset C_{b,1}(H)$  such that

$$\lim_{n \rightarrow \infty} \frac{\varphi_n}{1 + |\cdot|} \stackrel{\pi}{=} \frac{\varphi}{1 + |\cdot|}$$

we have, for any  $t \geq 0$ ,

$$\lim_{n \rightarrow \infty} \frac{P_t \varphi_n}{1 + |\cdot|} \stackrel{\pi}{=} \frac{P_t \varphi}{1 + |\cdot|}.$$

*Proof.* (i). Take  $\varphi \in C_{b,1}(H), t \geq 0$ . We have to show that  $P_t \varphi \in C_{b,1}(H)$ , that is the function  $x \mapsto (1 + |x|)^{-1} P_t \varphi(x)$  is uniformly continuous and bounded. Take  $\varepsilon > 0$  and let  $\theta_\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}$  be the modulus of continuity of  $(1 + |\cdot|)^{-1} \varphi$ . We have

$$\frac{P_t \varphi(x)}{1 + |x|} - \frac{P_t \varphi(y)}{1 + |y|} = I_1(t, x, y) + I_2(t, x, y) + I_3(t, x, y),$$

where

$$\begin{aligned} I_1(t, x, y) &= \mathbb{E} \left[ \left( \frac{\varphi(X(t, x))}{1 + |X(t, x)|} - \frac{\varphi(X(t, y))}{1 + |X(t, y)|} \right) \frac{1 + |X(t, x)|}{1 + |x|} \right], \\ I_2(t, x, y) &= \mathbb{E} \left[ \frac{\varphi(X(t, y))}{1 + |X(t, y)|} \left( \frac{|X(t, x)| - |X(t, y)|}{1 + |x|} \right) \right], \\ I_3(t, x, y) &= \mathbb{E} \left[ \frac{\varphi(X(t, y)) (1 + |X(t, x)|)}{1 + |X(t, y)|} \left( \frac{1}{1 + |x|} - \frac{1}{1 + |y|} \right) \right]. \end{aligned}$$

For  $I_1(t, x, y)$  we have, by taking into account (3), (4), that there exists  $c > 0$  such that

$$\begin{aligned} |I_1(t, x, y)| &\leq \mathbb{E} \left[ \theta_\varphi(|X(t, x) - X(t, y)|) \frac{1 + |X(t, x)|}{1 + |x|} \right] \\ &\leq \theta_\varphi(c|x - y|) \frac{\mathbb{E}[1 + |X(t, x)|]}{1 + |x|} \leq c\theta_\varphi(c|x - y|). \end{aligned}$$

Then, there exists  $\delta_1 > 0$  such that  $|I_1(t, x, y)| \leq \varepsilon/3$ , for any  $x, y \in H$  such that  $|x - y| \leq \delta_1$ . For  $I_2(t, x, y)$  we have, by elementary inequalities,

$$\begin{aligned} |I_2(t, x, y)| &\leq \frac{\|\varphi\|_{0,1}}{1 + |x|} \mathbb{E}[||X(t, x)| - |X(t, y)||] \\ &\leq \frac{\|\varphi\|_{0,1}}{1 + |x|} \mathbb{E}[|X(t, x) - X(t, y)|] \leq \|\varphi\|_{0,1} c|x - y|. \end{aligned}$$

Then there exists  $\delta_2 > 0$  such that  $|I_2(t, x, y)| \leq \varepsilon/3$ , for any  $x, y \in H$  such that  $|x - y| \leq \delta_2$ . Similarly, for  $I_3(t, x, y)$  we have

$$\begin{aligned} |I_3(t, x, y)| &\leq \|\varphi\|_{0,1} \frac{1 + \mathbb{E}[|X(t, x)|]}{1 + |x|} \frac{||x| - |y||}{1 + |y|} \\ &\leq c\|\varphi\|_{0,1}(1 + c)|x - y|. \end{aligned}$$

for some  $c > 0$ . Then, there exists  $\delta_3 > 0$  such that  $|I_3(t, x, y)| \leq \varepsilon/3$ , for any  $x, y \in H$  such that  $|x - y| \leq \delta_3$ . Finally, for any  $x, y \in H$  with  $|x - y| \leq \min\{\delta_1, \delta_2, \delta_3\}$  we find that

$$\left| \frac{P_t \varphi(x)}{1 + |x|} - \frac{P_t \varphi(y)}{1 + |y|} \right| < \varepsilon$$

as claimed. Now, by taking into account (4), there exists  $c > 0$  such that

$$\left| \frac{P_t \varphi(x)}{1 + |x|} \right| \leq \|\varphi\|_{0,1} \frac{1 + \mathbb{E}[|X(t, x)|]}{1 + |x|} \leq c\|\varphi\|_{0,1}$$



Then  $P_t\varphi \in C_{b,1}(H)$ . Note that by (4) it follows that the operators  $P_t$  are bounded in a neighborhood of 0. Hence, the existence of the two constants  $c_0 > 0$ ,  $\omega_0 \in \mathbb{R}$  follows by (iv) and by a standard argument. Notice that by the same argument follows<sup>1</sup> (v).

(ii). Take  $\varphi \in C_{b,1}(H)$ , and consider a sequence  $(\varphi_n)_{n \in \mathbb{N}} \subset C_b(H)$  such that

$$\lim_{n \rightarrow \infty} \frac{\varphi_n}{1 + |\cdot|} \stackrel{\pi}{=} \frac{\varphi}{1 + |\cdot|}. \quad (11)$$

Since  $\pi_t(t, \cdot)$  is the image measure of  $X(t, x)$  in  $H$ , the representation (ii) holds for any  $\varphi_n$ , that is

$$P_t\varphi_n(x) = \mathbb{E}[\varphi_n(X(t, x))] = \int_H \varphi_n(y) \pi_t(x, dy).$$

Since (4) holds we have  $\pi(x, \cdot) \in \mathcal{M}_1(H)$ , and by (11) there exists  $c > 0$  such that  $|\varphi_n(x)| \leq c(1 + |x|)$ , for any  $n \in \mathbb{N}$ ,  $x \in H$ . Finally, the result follows by the dominated convergence theorem.

(iii). For any  $\varphi \in C_{b,1}(H)$ ,  $x \in H$ ,  $t, s \geq 0$  we have

$$\begin{aligned} P_t\varphi(x) - P_s\varphi(x) &= \mathbb{E} \left[ \frac{\varphi(X(t, x))}{1 + |X(t, x)|} - \frac{\varphi(X(s, x))}{1 + |X(s, x)|} (1 + |X(t, x)|) \right] \\ &\quad + \mathbb{E} \left[ \frac{\varphi(X(s, x))}{1 + |X(s, x)|} (|X(t, x)| - |X(s, x)|) \right]. \end{aligned}$$

Then

$$\begin{aligned} |P_t\varphi(x) - P_s\varphi(x)| &\leq \mathbb{E} [\theta_\varphi (|X(t, x) - X(s, x)|) (1 + |X(t, x)|)] \\ &\quad + \|\varphi\|_{0,1} \mathbb{E} [|X(t, x) - X(s, x)|], \quad (12) \end{aligned}$$

where  $\theta_\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}$  is the modulus of continuity of  $(1 + |\cdot|)^{-1}\varphi$ . Note also that since for any  $x \in H$  the process  $(X(t, x))_{t \geq 0}$  is continuous in mean square, we have

$$\lim_{t \rightarrow s} |X(t, x) - X(s, x)| = 0 \quad \mathbb{P}\text{-a.s..}$$

Hence, by taking into account that  $\theta_\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}$  is bounded and that (4) holds, we can apply the dominated convergence theorem to show that the first term in the right-hand side of (12) vanishes as  $t \rightarrow s$ . Finally, the fact that the second term in the right-hand side of (12) vanishes as  $t \rightarrow s$  may be proved by the same argument.

(iv). Take  $\varphi \in C_{b,1}(H)$ , and consider a sequence  $(\varphi_n)_{n \in \mathbb{N}} \subset C_b(H)$  such

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<sup>1</sup>Of course, to prove (iv)-(v) we do not use this statement of (i)

that  $(1 + |\cdot|)^{-1}\varphi_n \xrightarrow{\pi} (1 + |\cdot|)^{-1}\varphi$  as  $n \rightarrow \infty$ . By the markovianity of the process  $X(t, x)$  it follows that (iv) holds true for any  $\varphi_n$ . Then, since by (iii) it follows that  $(1 + |\cdot|)^{-1}P_t\varphi_n \xrightarrow{\pi} (1 + |\cdot|)^{-1}P_t\varphi$  as  $n \rightarrow \infty$ , still by (iii) we find

$$\frac{P_{t+s}\varphi}{1 + |\cdot|} \stackrel{\pi}{=} \lim_{n \rightarrow \infty} \frac{P_{t+s}\varphi_n}{1 + |\cdot|} = \lim_{n \rightarrow \infty} \frac{P_t P_s \varphi_n}{1 + |\cdot|} \stackrel{\pi}{=} \frac{P_t P_s \varphi}{1 + |\cdot|}.$$

This concludes the proof.  $\square$

**Remark 2.3.** We recall that for any  $k > 0$ ,  $T > 0$  there exists  $c_k > 0$  such that

$$\sup_{t \in [0, T]} \mathbb{E}[|X(t, x)|^k] < c_k(1 + |x|^k),$$

that implies  $\{\pi_t(x, \cdot), t \geq 0, x \in H\} \subset \bigcap_{k \geq 0} \mathcal{M}_k(H)$ . Consequently, all the results of this section are true with  $C_{b,k}(H)$  replacing  $C_{b,1}(H)$ .

Here we collect some useful properties of the infinitesimal generator  $(K, D(K))$ .

**Proposition 2.4.** *Let  $X(t, x)$  be the mild solution of problem (1) and let  $(P_t)_{t \geq 0}$  be the associated transition semigroups in the space  $C_{b,1}(H)$  defined by (5). Let also  $(K, D(K))$  be the associated infinitesimal generators, defined by (6). Then*

- (i) *for any  $\varphi \in D(K)$ , we have  $P_t\varphi \in D(K)$  and  $KP_t\varphi = P_tK\varphi$ ,  $t \geq 0$ ;*
- (ii) *for any  $\varphi \in D(K)$ ,  $x \in H$ , the map  $[0, \infty) \rightarrow \mathbb{R}$ ,  $t \mapsto P_t\varphi(x)$  is continuously differentiable and  $(d/dt)P_t\varphi(x) = P_tK\varphi(x)$ ;*
- (iii) *given  $c_0 > 0$  and  $\omega_0$  as in Proposition 2.2, for any  $\lambda > \omega_0$  the linear operator  $R(\lambda, K)$  on  $C_{b,1}(H)$  done by*

$$R(\lambda, K)f(x) = \int_0^\infty e^{-\lambda t} P_t f(x) dt, \quad f \in C_{b,1}(H), x \in H$$

*satisfies, for any  $f \in C_{b,1}(H)$*

$$R(\lambda, K) \in \mathcal{L}(C_{b,1}(H)), \quad \|R(\lambda, K)\|_{\mathcal{L}(C_{b,1}(H))} \leq \frac{c_0}{\lambda - \omega_0}$$

$$R(\lambda, K)f \in D(K), \quad (\lambda I - K)R(\lambda, K)f = f.$$

*We call  $R(\lambda, K)$  the resolvent of  $K$  at  $\lambda$ .*

*Proof.* (i). It is proved by taking into account (6) and (iii) of Proposition 2.2.

(ii). This follows easily by (i) and by (iii) of Proposition 2.2.

(iii). By (i) of Proposition 2.2 we have

$$\left\| \int_0^\infty e^{-\lambda t} P_t f dt \right\|_{0,1} \leq c_0 \int_0^\infty e^{-(\lambda-\omega_0)t} dt \|f\|_{0,1} = \frac{c_0 \|f\|_{0,1}}{\lambda - \omega_0}.$$

Finally, the fact that  $R(\lambda, K)f \in D(K)$  and  $(\lambda I - K)R(\lambda, K)f = f$  hold can be proved in a standard way (see, for instance, [4], [10]).  $\square$

### 3 Proof of Theorem 1.2

In order to prove this theorem, we need some results about the transition semigroup  $(P_t)_{t \geq 0}$  in the space  $C_b(H)$ . Denote by  $\pi_t(x, \cdot)$  the image measure of the mild solution  $X(t, x)$  of problem (1). Since for any  $\varphi \in C_b(H)$  the representation

$$P_t \varphi(x) = \int_H \varphi(y) \pi_t(x, dy), \quad x \in H, t \geq 0$$

holds (cf (ii) of Proposition 2.2) and  $X(t, x)$  is continuous in mean square, it follows easily that  $(P_t)_{t \geq 0}$  is a semigroup of contraction operators in the space  $C_b(H)$ . Moreover, we have that for any  $x \in H, \varphi \in C_b(H)$  the function  $\mathbb{R}^+ \rightarrow \mathbb{R}, t \mapsto P_t \varphi(x)$  is continuous (cf (iii) of Proposition 2.2). This means that  $(P_t)_{t \geq 0}$  is *stochastically continuous Markov semigroup*, in the sense introduced in [9].

We denote by  $(K, D(K, C_b(H)))$  the infinitesimal generator of  $P_t$  in the space  $C_b(H)$ , defined by

$$\left\{ \begin{array}{l} D(K, C_b(H)) = \left\{ \varphi \in C_b(H) : \exists g \in C_b(H), \lim_{t \rightarrow 0^+} \frac{P_t \varphi(x) - \varphi(x)}{t} = g(x), \right. \\ \quad \left. x \in H, \sup_{t \in (0,1)} \left\| \frac{P_t \varphi - \varphi}{t} \right\|_0 < \infty \right\} \\ K\varphi(x) = \lim_{t \rightarrow 0^+} \frac{P_t \varphi(x) - \varphi(x)}{t}, \quad \varphi \in D(K, C_b(H)), x \in H. \end{array} \right. \quad (13)$$

It is clear that  $D(K, C_b(H)) \subset D(K)$ . The key result we use to prove the Theorem is the following, proved in [9]

**Theorem 3.1.** *For any  $\mu \in \mathcal{M}(H)$  there exists a unique family of measures  $\{\mu_t\}_{t \geq 0} \subset \mathcal{M}(H)$  such that*

$$\int_0^T |\mu_t|(H) dt < \infty, \quad \forall T > 0 \quad (14)$$

and

$$\int_H \varphi(x) \mu_t(dx) - \int_H \varphi(x) \mu(dx) = \int_0^t \left( \int_H K \varphi(x) \mu_s(dx) \right) ds \quad (15)$$

holds for any  $t \geq 0$ ,  $\varphi \in D(K, C_b(H))$ .

We split the proof into two lemmata.

**Lemma 3.2.** *The formula*

$$\langle \varphi, P_t^* F \rangle_{\mathcal{L}(C_{b,1}(H), (C_{b,1}(H))^*)} = \langle P_t \varphi, F \rangle_{\mathcal{L}(C_{b,1}(H), (C_{b,1}(H))^*)} \quad (16)$$

defines a semigroup of linear operators in  $(C_{b,1}(H))^*$ . Finally,  $P_t^* : \mathcal{M}_1(H) \rightarrow \mathcal{M}_1(H)$  and it maps positive measures into positive measures.

*Proof.* Fix  $t \geq 0$ . By (4) it follows that there exists  $c > 0$  such that  $|P_t \varphi(x)| \leq c \|\varphi\|_{0,1} (1 + |x|)$ , for any  $\varphi \in C_{b,1}(H)$ . Then, if  $F \in (C_{b,1}(H))^*$ , we have

$$|\langle \varphi, P_t^* F \rangle_{\mathcal{L}(C_{b,1}(H), (C_{b,1}(H))^*)}| \leq c \|F\|_{(C_{b,1}(H))^*} \|\varphi\|_{0,1},$$

for any  $\varphi \in C_{b,1}(H)$ . Since  $P_t^*$  is linear, it follows that  $P_t^* \in \mathcal{L}((C_{b,1}(H))^*)$ . Note that by (ii) of Proposition 2.2 it follows  $P_t \varphi \geq 0$  for any  $\varphi \geq 0$ . This implies that if  $\langle \varphi, F \rangle \geq 0$  for any  $\varphi \geq 0$ , then  $\langle \varphi, P_t^* F \rangle \geq 0$  for any  $\varphi \geq 0$ . Hence, in order to check that  $P_t^* : \mathcal{M}_1(H) \rightarrow \mathcal{M}_1(H)$ , it is sufficient to take  $\mu$  positive. So, let  $\mu \in \mathcal{M}_1(H)$  be positive and consider the map

$$\Lambda : \mathcal{B}(H) \rightarrow \mathbb{R}, \quad \Gamma \mapsto \Lambda(\Gamma) = \int_H \pi_t(x, \Gamma) \mu(dx).$$

We recall that since  $X(t, x)$  is continuous with respect to  $x$ , for any  $\Gamma \in \mathcal{B}(H)$  the map  $H \rightarrow [0, 1]$ ,  $x \rightarrow \pi_t(x, \Gamma)$  is Borel, and consequently the formula above is meaningful. It is straightforward to see that  $\Lambda$  is a positive and finite Borel measure on  $H$ , namely  $\Lambda \in \mathcal{M}(H)$ . We now show  $\Lambda = P_t^* \mu$ .

Let us fix  $\varphi \in C_b(H)$ , and consider a sequence of simple Borel functions  $(\varphi_n)_{n \in \mathbb{N}}$  which converges uniformly to  $\varphi$  and such that  $|\varphi_n(x)| \leq |\varphi(x)|$ ,  $x \in H$ . For any  $x \in H$  we have

$$\lim_{n \rightarrow \infty} \int_H \varphi_n(y) \pi_t(x, dy) = \int_H \varphi(y) \pi_t(x, dy) = P_t \varphi(x)$$

and

$$\sup_{x \in H} \left| \int_H \varphi_n(y) \pi_t(x, dy) \right| \leq \|\varphi\|_0.$$

Hence, by the dominated convergence theorem and by taking into account that  $\varphi_n$  is simple, we have

$$\begin{aligned} \int_H \varphi(x) \Lambda(dx) &= \lim_{n \rightarrow \infty} \int_H \varphi_n(x) \Lambda(dx) = \lim_{n \rightarrow \infty} \int_H \left( \int_H \varphi_n(y) \pi_t(x, dy) \right) \mu(dx) \\ &= \int_H \left( \int_H \varphi(y) \pi_t(x, dy) \right) \mu(dx) = \int_H P_t \varphi(x) \mu(dx). \end{aligned}$$

This implies that  $P_t^* \mu = \Lambda$  and consequently  $P_t^* \mu \in \mathcal{M}(H)$ .

In order to show that  $P_t^* \mu \in \mathcal{M}_1(H)$ , consider a sequence of functions  $(\psi_n)_{n \in \mathbb{N}} \subset C_b(H)$  such that  $\psi_n(x) \rightarrow |x|$  as  $n \rightarrow \infty$  and  $\psi(x) \leq |x|$ , for any  $x \in H$ . By Proposition 2.2 we have  $\int_H \psi_n(y) \pi_t(x, dy) \rightarrow \int_H |y| \pi_t(x, dy)$  as  $n \rightarrow \infty$  and  $\int_H \psi_n(y) \pi_t(x, dy) \leq c(1 + |x|)$ , for any  $x \in H$  and for some  $c > 0$ . Hence, since  $\mu \in \mathcal{M}_1(H)$  we have

$$\begin{aligned} \int_H |x| P_t^* \mu(dx) &= \lim_{n \rightarrow \infty} \int_H \psi_n(x) P_t^* \mu(dx) \\ &= \lim_{n \rightarrow \infty} \int_H \left( \int_H \psi_n(y) \pi_t(x, dy) \right) \mu(dx) \leq \int_H c(1 + |x|) \mu(dx) < \infty \end{aligned}$$

This concludes the proof.  $\square$

**Lemma 3.3.** *For any  $\mu \in \mathcal{M}_1(H)$  there exists a unique family of finite measures  $\{\mu_t\}_{t \geq 0} \subset \mathcal{M}_1(H)$  fulfilling (7), (8), and this family is given by  $\{P_t^* \mu\}_{t \geq 0}$ .*

*Proof.* We first check that  $\{P_t^* \mu\}_{t \geq 0}$  satisfies (7), (8). By Proposition 3.2, for any  $\mu \in \mathcal{M}_1(H)$ , formula (16) defines a family  $\{P_t^* \mu\}_{t \geq 0}$  of measures on  $H$ . Moreover, by (i) of Proposition 2.2 it follows that for any  $T > 0$  it holds

$$\sup_{t \in [0, T]} \|P_t^* \mu\|_{(C_{b,1}(H))^*} \sup_{t \in [0, T]} \int_H (1 + |x|) |P_t^* \mu|(dx) < \infty.$$

Hence, (7) holds. We now show (8). By (i), (ii), (iv) of Proposition 2.2 and by the dominated convergence theorem it follows easily that for any  $\varphi \in C_{b,1}(H)$  the function

$$\mathbb{R}^+ \rightarrow \mathbb{R}, \quad t \mapsto \int_H \varphi(x) P_t^* \mu(dx) \quad (17)$$

is continuous. Clearly,  $P_0^* \mu = \mu$ . Now we show that if  $\varphi \in D(K)$  then the function (17) is differentiable. Indeed, by taking into account (6) and (i) of

Proposition 2.4, for any  $\varphi \in D(K)$  we can apply the dominated convergence theorem to obtain

$$\begin{aligned}
\frac{d}{dt} \int_H \varphi(x) P_t^* \mu(dx) &= \\
&= \lim_{h \rightarrow 0} \frac{1}{h} \left( \int_H P_{t+h} \varphi(x) \mu(dx) - \int_H P_t \varphi(x) \mu_t(dx) \right) \\
&= \lim_{h \rightarrow 0} \int_H \left( \frac{P_{t+h} \varphi(x) - P_t \varphi(x)}{h} \right) \mu(dx) \\
&= \lim_{h \rightarrow 0} \int_H P_t \left( \frac{P_h \varphi - \varphi}{h} \right) (x) \mu(dx) \\
&= \int_H \lim_{h \rightarrow 0} \left( \frac{P_h \varphi - \varphi}{h} \right) (x) P_t^* \mu(dx) = \int_H K \varphi(x) P_t^* \mu(dx).
\end{aligned}$$

Then, by arguing as above, it follows that the differential of the mapping defined by (17) is continuous. This clearly implies that  $\{P_t^* \mu\}_{t \geq 0}$  satisfies (8). In order to show uniqueness of such a solution, by the linearity of the problem it is sufficient to show that if  $\mu = 0$  and  $\{\mu_t\}_{t \geq 0} \subset \mathcal{M}_1(H)$  is a solution of equation (8), then  $\mu_t = 0$ , for any  $t \geq 0$ . Note that equation (8) holds in particular for  $\varphi \in D(K, D(K))$  (cf (13)) and consequently (15) holds, for any  $\varphi \in D(K, D(K))$ . Note also that by (7) follows that  $\{\mu_t\}_{t \geq 0}$  fulfils (14). Hence, by Theorem 3.1, it follows that  $\mu_t = 0$ ,  $\forall t \geq 0$ . This concludes the proof.  $\square$

## 4 Proof of Theorem 1.3

We split the proof in several steps. We start by studying the Ornstein-Uhlenbeck operator in  $C_{b,1}(H)$  that is, roughly speaking, the case  $F = 0$  in (9). In Proposition 4.3 we shall prove Theorem 1.4 in the case  $F = 0$ . Then, Corollary 4.4 will show that  $(K, D(K_0))$  is an extension of  $K_0$  and  $K\varphi = K_0\varphi$  for any  $\varphi \in \mathcal{E}_A(H)$ . In order to complete the proof of the theorem, we shall present several approximation results. Finally, Lemma 4.6 will complete the proof.

## 4.1 The Ornstein-Uhlenbeck semigroup in $C_{b,1}(H)$

An important role in what follows it is played by the *Ornstein-Uhlenbeck semigroup*  $(R_t)_{t \geq 0}$  in the space  $C_{b,1}(H)$ , defined by the formula

$$R_t \varphi(x) = \begin{cases} \varphi(x), & t = 0, \\ \int_H \varphi(e^{tA}x + y) N_{Q_t}(dy), & t > 0 \end{cases}$$

where  $\varphi \in C_{b,1}(H)$ ,  $x \in H$  and  $N_{Q_t}$  is the gaussian measure of zero mean and covariance operator  $Q_t$  (cf Hypothesis 1.1). It is well known that the representation

$$R_t \varphi(x) = \mathbb{E} \left[ \varphi \left( e^{tA}x + \int_0^t e^{(t-s)A} B dW(s) \right) \right] \quad (18)$$

holds, for any  $t \geq 0$ ,  $\varphi \in C_{b,1}(H)$ ,  $x \in H$ . Hence, the Ornstein-Uhlenbeck semigroup  $(R_t)_{t \geq 0}$  is the transition semigroup (5) in the case  $F = 0$  in (1). Consequently,  $(R_t)_{t \geq 0}$  satisfies stamentes (i)–(v) of Proposition 2.2. It is well known the following identity

$$R_t(e^{i\langle \cdot, h \rangle})(x) = e^{i\langle e^{tA}x, h \rangle - \frac{1}{2}\langle Q_t h, h \rangle}, \quad (19)$$

which implies  $R_t : \mathcal{E}_A(H) \rightarrow \mathcal{E}_A(H)$ , for any  $t \geq 0$ . We define the infinitesimal generator  $L : D(L) \rightarrow C_{b,1}(H)$  of  $(R_t)_{t \geq 0}$  in  $C_{b,1}(H)$  as in (6), with  $L$  replacing  $K$  and  $R_t$  replacing  $P_t$ .

**Theorem 4.1.** *Let  $(P_t)_{t \geq 0}$  be the semigroup (5) and let  $(R_t)_{t \geq 0}$  be the Ornstein-Uhlenbeck semigroup (18). We denote by  $(K, D(K))$ ,  $(L, D(L))$  the corresponding infinitesimal generators in  $C_{b,1}(H)$ . Then we have  $D(L) \cap C_b^1(H) = D(K) \cap C_b^1(H)$  and  $K\varphi = L\varphi + \langle D\varphi, F \rangle$ , for any  $\varphi \in D(L) \cap C_b^1(H)$ .*

*Proof.* Let  $X(t, x)$  be the mild solution of equation (1) and let us set

$$Z_A(t, x) = e^{tA} + \int_0^t e^{(t-s)A} B dW(s).$$

Take  $\varphi \in D(L) \cap C_b^1(H)$ . By taking into account that

$$X(t, x) = Z_A(t, x) + \int_0^t e^{(t-s)A} F(X(s, x)) ds,$$

by the Taylor formula we have that  $\mathbb{P}$ -a.s. it holds

$$\varphi(Z_A(t, x)) = \varphi(Z_A(t, x)) - \varphi(X(t, x)) + \varphi(X(t, x))$$

$$= \varphi(X(t, x)) - \int_0^1 \left\langle D\varphi(\xi Z_A(t, x) + (1 - \xi)X(t, x)), \int_0^t e^{(t-s)A} F(X(s, x)) ds \right\rangle d\xi.$$

Then we have

$$R_t\varphi(x) - \varphi(x) = \mathbb{E}[\varphi(Z_A(t, x))] - \varphi(x) = P_t\varphi(x) - \varphi(x) \\ - \mathbb{E} \left[ \int_0^1 \left\langle D\varphi(\xi Z_A(t, x) + (1 - \xi)X(t, x)), \int_0^t e^{(t-s)A} F(X(s, x)) ds \right\rangle d\xi \right].$$

Since  $\varphi \in D(L) \cap C_b^1(H)$ , it follows easily that for any  $x \in H$

$$\lim_{t \rightarrow 0^+} \frac{P_t\varphi(x) - \varphi(x)}{t} = L\varphi(x) + \langle D\varphi(x), F(x) \rangle$$

and

$$\sup_{t \in (0, 1]} \left\| \frac{P_t\varphi - \varphi}{t} \right\|_{0,1} \\ \leq \sup_{t \in (0, 1]} \left\| \frac{R_t\varphi - \varphi}{t} \right\|_{0,1} + \sup_{x \in H} \|D\varphi(x)\|_{\mathcal{L}(H)} \sup_{x \in H} \frac{|F(x)|}{1 + |x|} < \infty,$$

that implies  $\varphi \in D(K)$  and  $K\varphi = L\varphi + \langle D\varphi, F \rangle$ . The opposite inclusion follows by interchanging the role of  $R_t$  and  $P_t$  in the Taylor formula.  $\square$

We need the following approximation result, proved in [9].

**Proposition 4.2.** *For any  $\varphi \in C_b(H)$ , there exists  $m \in \mathbb{N}$  and an  $m$ -indexed sequence  $(\varphi_{n_1, \dots, n_m})_{n_1, \dots, n_m \in \mathbb{N}} \subset \mathcal{E}_A(H)$  such that*

$$\lim_{n_1 \rightarrow \infty} \cdots \lim_{n_m \rightarrow \infty} \varphi_{n_1, \dots, n_m} \stackrel{\pi}{=} \varphi. \quad (20)$$

Moreover, if  $\varphi \in C_b^1(H)$ , we can choose the sequence in such a way that (20) holds and

$$\lim_{n_1 \rightarrow \infty} \cdots \lim_{n_m \rightarrow \infty} \langle D\varphi_{n_1, \dots, n_m}, h \rangle \stackrel{\pi}{=} \langle D\varphi, h \rangle,$$

for any  $h \in H$ .

Now we are able to prove the following

**Proposition 4.3.** *For any  $\varphi \in \mathcal{E}_A(H)$  we have  $\varphi \in D(L)$  and*

$$L\varphi(x) = \frac{1}{2} \text{Tr}[BB^* D^2\varphi(x)] + \langle x, A^* D\varphi(x) \rangle, \quad x \in H. \quad (21)$$



The set  $\mathcal{E}_A(H)$  is a  $\pi$ -core for  $(L, D(L))$ , and for any  $\varphi \in D(L) \cap C_b^1(H)$  there exists  $m \in \mathbb{N}$  and an  $m$ -indexed sequence  $(\varphi_{n_1, \dots, n_m})_{n_1, \dots, n_m \in \mathbb{N}} \subset \mathcal{E}_A(H)$  such that

$$\lim_{n_1 \rightarrow \infty} \cdots \lim_{n_m \rightarrow \infty} \frac{\varphi_{n_1, \dots, n_m}}{1 + |\cdot|} \stackrel{\pi}{=} \frac{\varphi}{1 + |\cdot|}, \quad (22)$$

$$\lim_{n_1 \rightarrow \infty} \cdots \lim_{n_m \rightarrow \infty} \frac{\frac{1}{2} \text{Tr}[BB^* D^2 \varphi_{n_1, \dots, n_m}] + \langle \cdot, A^* D \varphi_{n_1, \dots, n_m} \rangle}{1 + |\cdot|} \stackrel{\pi}{=} \frac{L\varphi}{1 + |\cdot|}. \quad (23)$$

Finally, if  $\varphi \in D(L) \cap C_b^1(H)$  we can choose the sequence in such a way that (22), (23) hold and

$$\lim_{n_1 \rightarrow \infty} \cdots \lim_{n_m \rightarrow \infty} \langle D \varphi_{n_1, \dots, n_m}, h \rangle \stackrel{\pi}{=} \langle D\varphi, h \rangle, \quad (24)$$

for any  $h \in H$ .

*Proof.* We recall that the proof of (21) may be found in [6], Remark 2.66 (in [6] the result is proved for the semigroup  $(R_t)_{t \geq 0}$  in the space  $C_{b,2}(H)$ , but it is clear that the result holds also in  $C_{b,1}(H)$ ).

Here we give only a sketch of the proof, which is very similar to the proof given in [9]. Take  $\varphi \in D(L)$ . For any  $n_2 \in \mathbb{N}$ , set  $\varphi_{n_2}(x) = n_2 \varphi(x) / (n_2 + |x|^2)$ . Clearly,  $\varphi_{n_2} \in C_b(H)$  and  $(1 + |\cdot|)^{-1} \varphi_{n_2} \xrightarrow{\pi} (1 + |\cdot|)^{-1} \varphi$  as  $n_2 \rightarrow \infty$ . By Proposition 4.2, for any  $n_2 \in \mathbb{N}$  we fix a sequence<sup>2</sup>  $(\varphi_{n_2, n_3})_{n_3 \in \mathbb{N}} \subset \mathcal{E}_A(H)$  such that  $\varphi_{n_2, n_3} \xrightarrow{\pi} \varphi_{n_2}$  as  $n_3 \rightarrow \infty$ . Set now, for any  $n_1, n_2, n_3, n_4 \in \mathbb{N}$

$$\varphi_{n_1, n_2, n_3, n_4}(x) = \frac{1}{n_4} \sum_{k=1}^{n_4} R_{\frac{k}{n_1 n_4}} \varphi_{n_2, n_3}(x). \quad (25)$$

Since for any  $\varphi \in C_{b,1}(H)$ ,  $x \in H$  the function  $\mathbb{R}^+ \rightarrow \mathbb{R}$ ,  $t \mapsto R_t \varphi(x)$  is continuous, a straightforward computation shows that the sequence  $(\varphi_{n_1, \dots, n_4})$  fulfils (22). Similarly, we find that for any  $x \in H$  it holds

$$\begin{aligned} & \lim_{n_1 \rightarrow \infty} \lim_{n_2 \rightarrow \infty} \lim_{n_3 \rightarrow \infty} \lim_{n_4 \rightarrow \infty} \frac{1}{2} \text{Tr}[BB^* D^2 \varphi_{n_1, n_2, n_3, n_4}(x)] + \langle x, A^* D \varphi_{n_1, n_2, n_3, n_4}(x) \rangle \\ &= \lim_{n_1 \rightarrow \infty} \lim_{n_2 \rightarrow \infty} \lim_{n_3 \rightarrow \infty} \lim_{n_4 \rightarrow \infty} L \varphi_{n_1, n_2, n_3, n_4}(x) \\ &= \lim_{n_1 \rightarrow \infty} \lim_{n_2 \rightarrow \infty} \lim_{n_3 \rightarrow \infty} n_1 \int_0^{\frac{1}{n_1}} L R_t \varphi_{n_2, n_3}(x) dt \\ &= \lim_{n_1 \rightarrow \infty} \lim_{n_2 \rightarrow \infty} \lim_{n_3 \rightarrow \infty} n_1 \left( R_{\frac{1}{n_1}} \varphi_{n_2, n_3}(x) - \varphi_{n_2, n_3}(x) \right) \\ &= \lim_{n_1 \rightarrow \infty} \left( R_{\frac{1}{n_1}} \varphi(x) - \varphi(x) \right) = L \varphi(x). \end{aligned}$$

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<sup>2</sup>we assume that the sequence has only one index

Here we have used the continuity of  $t \mapsto LR_t\varphi_{n_2,n_3}(x)$  and the fact that  $LR_t\varphi_{n_2,n_3}(x) = (d/dt)R_t\varphi_{n_2,n_3}(x)$  (cf Proposition 2.2 and Proposition 2.4). The fact that any limit above is equibounded in  $C_{b,1}(H)$  with respect to the corresponding index follows by the construction of  $\varphi_{n_1,n_2,n_3,n_4}(x)$ . Hence, (23) follows.

If  $\varphi \in D(L) \cap C_b^1(H)$ , by Proposition 4.2, there exists a sequence<sup>3</sup>  $(\varphi_n)_{n \in \mathbb{N}} \subset \mathcal{E}_A(H)$  such that  $\varphi_n \xrightarrow{\pi} \varphi$  as  $n \rightarrow \infty$  and  $\langle D\varphi_n, h \rangle \xrightarrow{\pi} \langle D\varphi, h \rangle$  as  $n \rightarrow \infty$ , for any  $h \in H$ . Since for any  $t > 0$ ,  $n \in \mathbb{N}$  we have

$$\langle DR_t\varphi_n(x), h \rangle = \int_H \langle D\varphi_n(e^{tA}x + y), h \rangle N_{Q_t}(dy), \quad x \in H$$

it follows  $\langle DR_t\varphi_n, h \rangle \xrightarrow{\pi} \langle DR_t\varphi, h \rangle$  as  $n \rightarrow \infty$ , for any  $h \in H$ . Then, the claim follows by arguing as above.  $\square$

By Theorem 4.1 and Proposition 4.3 we have

**Corollary 4.4.**  *$(K, D(K))$  is an extension of  $K_0$  and for any  $\varphi \in \mathcal{E}_A(H)$  we have  $\varphi \in D(K)$  and  $K\varphi = K_0\varphi$ .*

## 4.2 Approximation of $F$ with smooth functions

It is convenient to introduce an auxiliary Ornstein–Uhlenbeck semigroup

$$U_t\varphi(x) = \int_H \varphi(e^{tS}x + y) N_{\frac{1}{2} S^{-1}(e^{2tS}-1)}(dy), \quad \varphi \in C_b(H)$$

where  $S : D(S) \subset H \rightarrow H$  is a self-adjoint negative definite operator such that  $S^{-1}$  is of trace class. We notice that  $U_t$  is strong Feller, and for any  $t > 0$ ,  $\varphi \in C_b(H)$ ,  $U_t\varphi$  is infinite times differentiable with bounded differentials (see [6]). We introduce a regularization of  $F$  by setting

$$\langle F_n(x), h \rangle = \int_H \left\langle F\left(e^{\frac{1}{n}S}x + y\right), e^{\frac{1}{n}S}h \right\rangle N_{\frac{1}{2} S^{-1}(e^{\frac{2}{n}S}-1)}(dy), \quad n \in \mathbb{N}.$$

It is easy to check that  $F_n$  is infinite times differentiable, with first differential bounded by  $\kappa$ , for any  $n \in \mathbb{N}$ . Moreover,  $F_n(x) \rightarrow F(x)$  as  $n \rightarrow \infty$  for all  $x \in H$  and  $|F_n(x)| \leq |F(x)|$ , for all  $n \in \mathbb{N}$ ,  $x \in H$ .

Let  $P_t^n$  be the transition semigroup

$$P_t^n\varphi(x) = \mathbb{E}[\varphi(X^n(t, x))], \quad \varphi \in C_{b,1}(H) \tag{26}$$

---

<sup>3</sup>we assume that the sequence has only one index

where  $X^n(t, x)$  is the solution of (1) with  $F_n$  replacing  $F$ . It is easy to check

$$\lim_{n \rightarrow \infty} \mathbb{E}[|X^n(t, x) - X(t, x)|^2] = 0, \quad t \geq 0, x \in H$$

and

$$\mathbb{E}[|X^n(t, x)|] \leq \mathbb{E}[|X(t, x)|], \quad t \geq 0, x \in H,$$

where  $c_0 > 0$ ,  $\omega_0 \in \mathbb{R}$  are as in Proposition 2.2. This implies

$$\lim_{n \rightarrow \infty} \frac{P_t^n \varphi}{1 + |\cdot|} \stackrel{\pi}{=} \frac{P_t \varphi}{1 + |\cdot|}, \quad (27)$$

for any  $t \geq 0$ ,  $\varphi \in C_{b,1}(H)$ . We denote by  $(K_n, D(K_n))$  the infinitesimal generator of the semigroup  $P_t^n$  in  $C_{b,1}(H)$ , defined as in (6) with  $K_n$  replacing  $K$  and  $P_t^n$  replacing  $P_t$ . We recall that all the statements of Proposition 2.2, Theorem 3.1 hold also for  $P_t^n$  and  $(K_n, D(K_n))$ . We also recall that the resolvent of  $(K, D(K))$  in  $C_{b,1}(H)$  is defined for any  $\lambda > \omega_0$  by the formula  $R(\lambda, K)f(x) = \int_0^\infty e^{-\lambda t} P_t f(x) dt$ ,  $f \in C_{b,1}(H)$ ,  $x \in H$  (cf Theorem 3.1). Similarly, for a fixed  $n \in \mathbb{N}$  the resolvent of  $(K_n, D(K_n))$  in  $C_{b,1}(H)$  at  $\lambda > 0$  is defined by the same formula with  $P_t^n$  replacing  $P_t$ . Since (27) holds, it is straightforward to see that

$$\lim_{n \rightarrow \infty} \frac{R(\lambda, K_n)\varphi}{1 + |\cdot|} \stackrel{\pi}{=} \frac{R(\lambda, K)\varphi}{1 + |\cdot|}, \quad (28)$$

for any  $\varphi \in C_{b,1}(H)$ ,  $\lambda > \omega_0$ .

The following proposition follows by Corollary 4.9 of [9] and by the fact that  $\|DF_n\| \leq \kappa$ , for any  $n \in \mathbb{N}$ .

**Proposition 4.5.** *For any  $n \in \mathbb{N}$ , let  $(K_n, D(K_n))$  be the infinitesimal generator of the semigroup (26). Then for any  $\lambda > \max\{0, \omega + M\kappa\}$ , the resolvent  $R(\lambda, K_n)$  of  $K_n$  at  $\lambda$  maps  $C_b^1$  into  $C_b^1(H)$  and it holds*

$$\|DR(\lambda, K_n)f\|_{C_b(H;H)} \leq \frac{M\|Df\|_{C_b(H;H)}}{\lambda - (\omega + M\kappa)}, \quad f \in C_b^1(H). \quad (29)$$

Corollary 4.4 shows that  $K$  is an extension of  $K_0$  and that  $K\varphi = K_0\varphi$ ,  $\forall \varphi \in \mathcal{E}_A(H)$ . So, in view of the fact that  $KP_t\varphi = P_tK_0\varphi$  for any  $\varphi \in \mathcal{E}_A(H)$  (cf (i) of Proposition 2.4), it is not difficult to check that  $\{P_t^*\mu\}_{t \geq 0}$  fulfils (7), (10). Now, let  $\mu \in \mathcal{M}_1(H)$  and assume that  $\{\mu_t\}_{t \geq 0} \subset \mathcal{M}_1(H)$  fulfils (7), (10). In view of Theorem 3.3, to prove that  $\mu_t = P_t^*\mu$ , for any  $t \geq 0$ , it is sufficient to show that  $\{\mu_t\}_{t \geq 0}$  is also a solution of (8). In order to do this, we need an approximation result.

**Lemma 4.6.** *The set  $\mathcal{E}_A(H)$  is a  $\pi$ -core for  $(K, D(K))$ , and for any  $\varphi \in D(K)$  there exist  $m \in \mathbb{N}$  and an  $m$ -indexed sequence  $(\varphi_{n_1, \dots, n_m}) \subset \mathcal{E}_A(H)$  such that*

$$\lim_{n_1 \rightarrow \infty} \cdots \lim_{n_m \rightarrow \infty} \frac{\varphi_{n_1, \dots, n_m}}{1 + |\cdot|} \stackrel{\pi}{=} \frac{\varphi}{1 + |\cdot|}, \quad (30)$$

$$\lim_{n_1 \rightarrow \infty} \cdots \lim_{n_m \rightarrow \infty} \frac{K_0 \varphi_{n_1, \dots, n_m}}{1 + |\cdot|} \stackrel{\pi}{=} \frac{K\varphi}{1 + |\cdot|}. \quad (31)$$

*Proof. Step 1.* Let<sup>4</sup>  $\varphi \in D(K)$ ,  $\lambda > \max\{0, \omega_0, \omega + M\kappa\}$  and set  $f = \lambda\varphi - K\varphi$ . We fix a sequence  $(f_{n_1}) \subset C_b^1(H)$  such that

$$\lim_{n_1 \rightarrow \infty} \frac{f_{n_1}}{1 + |\cdot|} \stackrel{\pi}{=} \frac{f}{1 + |\cdot|}.$$

Set  $\varphi_{n_1} = R(\lambda, K)f_{n_1}$ . By Proposition 2.4 it follows

$$\lim_{n_1 \rightarrow \infty} \frac{\varphi_{n_1}}{1 + |\cdot|} \stackrel{\pi}{=} \frac{\varphi}{1 + |\cdot|}, \quad \lim_{n_1 \rightarrow \infty} \frac{K\varphi_{n_1}}{1 + |\cdot|} \stackrel{\pi}{=} \frac{K\varphi}{1 + |\cdot|}. \quad (32)$$

**Step 2.** Now set  $\varphi_{n_1, n_2} = R(\lambda, K_{n_2})f_{n_1}$ , where  $K_{n_2}$  is the infinitesimal generator of the semigroup  $P_t^{n_2}$ , introduced in (26). Since  $f_{n_1} \in C_b^1(H)$ , by Proposition 4.5 we have  $\varphi_{n_1, n_2} \in C_b^1(H)$  and

$$\sup_{n_2 \in \mathbb{N}} \|D\varphi_{n_1, n_2}\|_{C_b(H; H)} \leq \frac{M\|Df_{n_1}\|_{C_b(H; H)}}{\lambda - (\omega + M\kappa)}, \quad (33)$$

for any  $n_1 \in \mathbb{N}$ . Moreover, by (28) it holds

$$\lim_{n_2 \rightarrow \infty} \varphi_{n_1, n_2} \stackrel{\pi}{=} \varphi_{n_1}, \quad \lim_{n_2 \rightarrow \infty} K_{n_2} \varphi_{n_1, n_2} \stackrel{\pi}{=} K\varphi_{n_1}, \quad (34)$$

for any  $n_1 \in \mathbb{N}$ . Since  $\varphi_{n_1, n_2} \in D(K_{n_2}) \cap C_b^1(H)$ , by Theorem 4.1 we have

$$K_{n_2} \varphi_{n_1, n_2} = L\varphi_{n_1, n_2} + \langle D\varphi_{n_1, n_2}, F_{n_2} \rangle.$$

**Step 3.** By Proposition 4.3, for any  $n_1, n_2$  there exists a sequence  $(\varphi_{n_1, n_2, n_3}) \subset \mathcal{E}_A(H)$  (we still assume that it has only one index) such that

$$\lim_{n_3 \rightarrow \infty} \varphi_{n_1, n_2, n_3} \stackrel{\pi}{=} \varphi_{n_1, n_2}, \quad \lim_{n_3 \rightarrow \infty} \frac{L\varphi_{n_1, n_2, n_3}}{1 + |\cdot|} \stackrel{\pi}{=} \frac{L\varphi_{n_1, n_2}}{1 + |\cdot|} \quad (35)$$

and

$$\lim_{n_3 \rightarrow \infty} \langle D\varphi_{n_1, n_2, n_3}, h \rangle \stackrel{\pi}{=} \langle D\varphi_{n_1, n_2}, h \rangle.$$

---

<sup>4</sup>the assumption  $\lambda > \omega_0$  is necessary to define the resolvent of  $K$  (cf Proposition 2.4)

for any  $h \in H$ . Notice that the since  $F_{n_2}$  is globally Lipschitz, it follows

$$\lim_{n_3 \rightarrow \infty} \frac{\langle D\varphi_{n_1, n_2, n_3}, F_{n_2} \rangle}{1 + |\cdot|} \stackrel{\pi}{=} \frac{\langle D\varphi_{n_1, n_2}, F_{n_2} \rangle}{1 + |\cdot|}.$$

This, together with (35), implies that the sequence  $(\varphi_{n_1, n_2, n_3})$  fulfils

$$\lim_{n_3 \rightarrow \infty} \varphi_{n_1, n_2, n_3} \stackrel{\pi}{=} \varphi_{n_1, n_2}, \quad \lim_{n_3 \rightarrow \infty} \frac{K_{n_2} \varphi_{n_1, n_2, n_3}}{1 + |\cdot|} \stackrel{\pi}{=} \frac{K_{n_2} \varphi_{n_1, n_2}}{1 + |\cdot|}.$$

Since  $K$  is an extension of  $K_0$  (cf Corollary 4.4), we have

$$K\varphi_{n_1, n_2, n_3} = K_0\varphi_{n_1, n_2, n_3} = K_{n_2}\varphi_{n_1, n_2, n_3} + \langle D\varphi_{n_1, n_2, n_3}, F - F_{n_2} \rangle$$

for any  $n_1, n_2, n_3 \in \mathbb{N}$ . So we find

$$\lim_{n_3 \rightarrow \infty} \frac{K_0\varphi_{n_1, n_2, n_3}}{1 + |\cdot|} \stackrel{\pi}{=} \frac{K_{n_2}\varphi_{n_1, n_2} + \langle D\varphi_{n_1, n_2}, F - F_{n_2} \rangle}{1 + |\cdot|}, \quad (36)$$

for any  $n_1, n_2 \in \mathbb{N}$ . Moreover, by (33), we see that

$$\frac{|\langle D\varphi_{n_1, n_2}(x), F(x) - F_{n_2}(x) \rangle|}{1 + |x|} \leq \frac{M\|Df_{n_1}\|_{C_b(H; H)}}{\lambda - (\omega + M\kappa)} \frac{|F(x) - F_{n_2}(x)|}{1 + |x|}$$

and consequently

$$\lim_{n_2 \rightarrow \infty} \frac{\langle D\varphi_{n_1, n_2}, F - F_{n_2} \rangle}{1 + |\cdot|} \stackrel{\pi}{=} 0.$$

This, together with (34) implies

$$\lim_{n_2 \rightarrow \infty} \frac{K_{n_2}\varphi_{n_1, n_2} + \langle D\varphi_{n_1, n_2}, F - F_{n_2} \rangle}{1 + |\cdot|} \stackrel{\pi}{=} \frac{K\varphi_{n_1}}{1 + |\cdot|}. \quad (37)$$

Finally, by taking into account (32), (36); (37), the sequence  $(\varphi_{n_1, n_2, n_3}) \subset \mathcal{E}_A(H)$  fulfils

$$\begin{aligned} \lim_{n_1 \rightarrow \infty} \lim_{n_2 \rightarrow \infty} \lim_{n_3 \rightarrow \infty} \frac{\varphi_{n_1, n_2, n_3}}{1 + |\cdot|} &\stackrel{\pi}{=} \frac{\varphi}{1 + |\cdot|}, \\ \lim_{n_1 \rightarrow \infty} \lim_{n_2 \rightarrow \infty} \lim_{n_3 \rightarrow \infty} \frac{K_0\varphi_{n_1, n_2, n_3}}{1 + |\cdot|} &\stackrel{\pi}{=} \frac{K\varphi}{1 + |\cdot|}. \end{aligned}$$

This concludes the proof.  $\square$

## 5 Proof of Theorem 1.4

Let  $\varphi \in D(K)$  and assume that  $(\varphi_n)_{n \in \mathbb{N}} \subset \mathcal{E}_A(H)$  fulfils (30), (31) (for simplicity we assume that this sequence has only one index: this does not change the generality of the proof). For any  $t \geq 0$  we find

$$\begin{aligned} \int_H \varphi(x) \mu_t(dx) - \int_H \varphi(x) \mu(dx) &= \lim_{n \rightarrow \infty} \left( \int_H \varphi_n(x) \mu_t(dx) - \int_H \varphi_n(x) \mu(dx) \right) \\ &= \lim_{n \rightarrow \infty} \int_0^t \left( \int_H K_0 \varphi_n(x) \mu_s(dx) \right) ds, \end{aligned}$$

since  $\varphi_n \in D(K)$  and  $K\varphi_n = K_0\varphi_n$ , for any  $n \in \mathbb{N}$  (cf Corollary 4.4). Now observe that since  $\sup_{n \in \mathbb{N}} |K_0\varphi_n(x)| \leq c(1 + |x|)$  for some  $c > 0$  and since  $\mu_s \in \mathcal{M}_1(H)$  for any  $s \geq 0$ , it holds

$$\lim_{n \rightarrow \infty} \int_H K_0 \varphi_n(x) \mu_s(dx) = \int_H K\varphi(x) \mu_s(dx)$$

and

$$\sup_{n \in \mathbb{N}} \left| \int_H K_0 \varphi_n(x) \mu_s(dx) \right| \leq c \int_H (1 + |x|) |\mu_s|(dx).$$

Hence, by taking into account (7) we can apply the dominated convergence theorem to obtain

$$\lim_{n \rightarrow \infty} \int_0^t \left( \int_H K_0 \varphi_n(x) \mu_s(dx) \right) ds = \int_0^t \left( \int_H K\varphi(x) \mu_s(dx) \right) ds$$

So,  $\{\mu_t\}_{t \geq 0}$  is also a solution of the measure equation for  $(K, D(K))$ . Since by Theorem 1.2 such a solution is unique and it is given by  $\{P_t^* \mu\}_{t \geq 0}$ , we have  $\int_H \varphi(x) P_t^* \mu(dx) = \int_H \varphi(x) \mu_t(dx)$ , for any  $\varphi \in \mathcal{E}_A(H)$ . By taking into account that the set  $\mathcal{E}_A(H)$  is  $\pi$ -dense in  $C_b(H)$  (cf. Proposition 4.2), we have  $\int_H \varphi(x) P_t^* \mu(dx) = \int_H \varphi(x) \mu_t(dx)$ , for any  $\varphi \in C_b(H)$ . this clearly implies  $P_t^* \mu = \mu_t$ ,  $\forall t \geq 0$ . This concludes the proof.

## 6 The reaction-diffusion case

We shall consider here the stochastic heat equation perturbed by a polynomial term of odd degree  $d > 1$  having negative leading coefficient (this will ensures non explosion). We shall represent this polynomial as

$$\lambda \xi - p(\xi), \quad \xi \in \mathbb{R},$$

where  $\lambda \in \mathbb{R}$  and  $p$  is an increasing polynomial, that is  $p'(\xi) \geq 0$  for all  $\xi \in \mathbb{R}$ .

We set  $H = L^2(\mathcal{O})$  where  $\mathcal{O} = [0, 1]^n$ ,  $n \in \mathbb{N}$ , and denote by  $\partial\mathcal{O}$  the boundary of  $\mathcal{O}$ . We are concerned with the following stochastic differential equation with Dirichlet boundary conditions

$$\begin{cases} dX(t, \xi) = [\Delta_\xi X(t, \xi) + \lambda X(t, \xi) - p(X(t, \xi))]dt + BdW(t, \xi), & \xi \in \mathcal{O}, \\ X(t, \xi) = 0, & t \geq 0, \xi \in \partial\mathcal{O}, \\ X(0, \xi) = x(\xi), & \xi \in \mathcal{O}, x \in H, \end{cases} \quad (38)$$

where  $\Delta_\xi$  is the Laplace operator,  $B \in \mathcal{L}(H)$  and  $W$  is a cylindrical Wiener process in a stochastic basis  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$  in  $H$ . We choose  $W$  of the form

$$W(t, \xi) = \sum_{k=1}^{\infty} e_k(\xi) \beta_k(t), \quad \xi \in \mathcal{O}, t \geq 0,$$

where  $\{e_k\}$  is a complete orthonormal system in  $H$  and  $\{\beta_k\}$  a sequence of mutually independent standard Brownian motions on  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ .

Let us write problem (38) as a stochastic differential equation in the Hilbert space  $H$ . For this we denote by  $A$  the realization of the Laplace operator with Dirichlet boundary conditions,

$$\begin{cases} Ax = \Delta_\xi x, & x \in D(A), \\ D(A) = H^2(\mathcal{O}) \cap H_0^1(\mathcal{O}). \end{cases} \quad (39)$$

$A$  is self-adjoint and has a complete orthonormal system of eigenfunctions, namely

$$e_k(\xi) = (2/\pi)^{n/2} \sin(\pi k_1 \xi_1) \cdots (\sin \pi k_n \xi_n),$$

where  $k = (k_1, \dots, k_n)$ ,  $k_i \in \mathbb{N}$ . For any  $x \in H$  we set  $x_k = \langle x, e_k \rangle$ ,  $k \in \mathbb{N}^n$ . Notice that

$$Ae_k = -\pi^2 |k|^2, \quad k \in \mathbb{N}^n, |k|^2 = k_1^2 + \cdots + k_n^2.$$

Therefore, we have

$$\|e^{tA}\| \leq e^{-\pi^2 t}, \quad t \geq 0. \quad (40)$$

**Remark 6.1.** We can also consider the realization of the Laplace operators with Neumann boundary conditions

$$\begin{cases} Nx = \Delta_\xi x, & x \in D(N), \\ D(N) = \left\{ x \in H^2(\mathcal{O}) : \frac{\partial x}{\partial \eta} = 0 \text{ on } \partial\mathcal{O} \right\} \end{cases}$$

where  $\eta$  represents the outward normal to  $\partial\mathcal{O}$ . Then

$$Nf_k = -\pi^2|k|^2 f_k, \quad k \in (\mathbb{N} \cap \{0\})^n,$$

where

$$f_k(\xi) = (2/\pi)^{n/2} \cos(\pi k_1 \xi_1) \cdots (\cos \pi k_n \xi_n),$$

$k = (k_1, \dots, k_n)$ ,  $k_i \in \mathbb{N} \cup \{0\}$  and  $|k|^2 = k_1^2 + \cdots + k_n^2$ .

Concerning the operator  $B$  we shall assume, for the sake of simplicity <sup>(5)</sup>, that  $B = (-A)^{-\gamma/2}$ , where

$$\gamma > \frac{n}{2} - 1. \quad (41)$$

Now, setting  $X(t) = X(t, \cdot)$  and  $W(t) = W(t, \cdot)$ , we shall write problem (38) as

$$\begin{cases} dX(t) = [AX(t) + F(X(t))]dt + (-A)^{-\gamma/2}dW(t), \\ X(0) = x. \end{cases} \quad (42)$$

where  $F$  is the mapping

$$F : D(F) = L^{2d}(\mathcal{O}) \subset H \rightarrow H, \quad x(\xi) \mapsto \lambda\xi - p(x(\xi)).$$

It is well known that for any  $x \in L^{2d}(\mathcal{O})$  problem (42) has a unique mild solution  $(X(t, x))_{t \geq 0, x \in H}$  (see, for instance, [5], [6]), fulfilling

$$X(t, x) = e^{tA}x + \int_0^t e^{(t-s)A} B dW(s) + \int_0^t e^{(t-s)A} F(X(s, x)) ds \quad (43)$$

for any  $t \geq 0$ . Finally, it is well known that for any  $T > 0$  there exists  $c > 0$  such that

$$\sup_{t \in [0, T]} \mathbb{E} \left[ |X(t, x)|_{L^{2d}(\mathcal{O})}^d \right] \leq c \left( 1 + |x|_{L^{2d}(\mathcal{O})}^d \right). \quad (44)$$

$$|X(t, x) - X(t, y)| \leq e^{(\lambda - \pi^2)t} |x - y|, \quad (45)$$

see [6, Theorem 4.8].

## 6.1 Main results

We consider here the Kolmogorov operator

$$K_0\varphi(x) = \frac{1}{2} \text{Tr}[BB^* D^2\varphi(x)] + \langle x, A^* D\varphi(x) \rangle + \langle D\varphi(x), F(x) \rangle, \quad x \in L^{2d}(\mathcal{O}). \quad (46)$$

---

<sup>5</sup> All following results remain true taking  $B = G(-A)^{-\gamma/2}$  with  $G \in \mathcal{L}(H)$ .



We are interested in extending the results of Theorems 1.2, 1.3, 1.4 to this operator. This will be done in Theorems 6.2, 6.3, 6.4 respectively.

Denote by  $C_{b,d}(L^{2d}(\mathcal{O}))$  the space of all functions  $\varphi : L^{2d}(\mathcal{O}) \rightarrow \mathbb{R}$  such that the function

$$L^{2d}(\mathcal{O}) \rightarrow \mathbb{R}, \quad x \rightarrow \frac{\varphi(x)}{1 + |x|_{L^{2d}(\mathcal{O})}^d}$$

is uniformly continuous and bounded. The space  $C_{b,d}(L^{2d}(\mathcal{O}))$ , endowed with the norm

$$\|\varphi\|_{C_{b,d}(L^{2d}(\mathcal{O}))} = \sup_{x \in L^{2d}(\mathcal{O})} \frac{|\varphi(x)|}{1 + |x|_{L^{2d}(\mathcal{O})}^d}$$

is a Banach space. Thanks to estimates (44) and (45) we can define a semigroup of transitional operators in  $C_{b,d}(L^{2d}(\mathcal{O}))$ , by the formula

$$P_t \varphi(x) = \mathbb{E}[\varphi(X(t, x))], \quad t \geq 0, \varphi \in C_{b,d}(L^{2d}(\mathcal{O})), x \in L^{2d}(\mathcal{O}), \quad (47)$$

see Proposition 6.5. We define its infinitesimal generator by setting

$$\left\{ \begin{aligned} D(K) &= \left\{ \varphi \in C_{b,d}(L^{2d}(\mathcal{O})) : \exists g \in C_{b,d}(L^{2d}(\mathcal{O})), \lim_{t \rightarrow 0^+} \frac{P_t \varphi(x) - \varphi(x)}{t} = \right. \\ &\quad \left. = g(x), x \in L^{2d}(\mathcal{O}), \sup_{t \in (0,1)} \left\| \frac{P_t \varphi - \varphi}{t} \right\|_{C_{b,d}(L^{2d}(\mathcal{O}))} < \infty \right\} \\ K\varphi(x) &= \lim_{t \rightarrow 0^+} \frac{P_t \varphi(x) - \varphi(x)}{t}, \quad \varphi \in D(K), x \in L^{2d}(\mathcal{O}). \end{aligned} \right. \quad (48)$$

Let  $\mathcal{M}_d(L^{2d}(\mathcal{O}))$  be the space of all Borel finite measures on  $L^{2d}(\mathcal{O})$  such that

$$\int_{L^{2d}(\mathcal{O})} |x|_{L^{2d}(\mathcal{O})}^d |\mu|(dx) < \infty.$$

Since  $L^{2d}(\mathcal{O}) \subset H$ , we have  $\mathcal{M}_d(L^{2d}(\mathcal{O})) \subset \mathcal{M}(H)$ . The following theorem generalize Theorem 1.2 to the reaction-diffusion case.

**Theorem 6.2.** *Let  $(P_t)_{t \geq 0}$  be the semigroup defined by (47)  $C_{b,2}(H)$ , and let  $(K, D(K))$  be its infinitesimal generator in  $C_{b,d}(L^{2d}(\mathcal{O}))$ , defined by (48). Then, the formula*

$$\langle \varphi, P_t^* F \rangle_{\mathcal{L}(C_{b,d}(L^{2d}(\mathcal{O})), (C_{b,d}(L^{2d}(\mathcal{O})))^*)} = \langle P_t \varphi, F \rangle_{\mathcal{L}(C_{b,d}(L^{2d}(\mathcal{O})), (C_{b,d}(L^{2d}(\mathcal{O})))^*)}$$

*defines a semigroup  $(P_t^*)_{t \geq 0}$  of linear and continuous operators on  $(C_{b,d}(L^{2d}(\mathcal{O})))^*$  that maps  $\mathcal{M}_d(L^{2d}(\mathcal{O}))$  into  $\mathcal{M}_d(L^{2d}(\mathcal{O}))$ . Moreover, for any  $\mu \in \mathcal{M}_d(L^{2d}(\mathcal{O}))$*

there exists a unique family of measures  $\{\mu_t\}_{t \geq 0} \subset \mathcal{M}_d(L^{2d}(\mathcal{O}))$  such that

$$\int_0^T \left( \int_H |x|_{L^{2d}(\mathcal{O})}^d |\mu_t|(dx) \right) dt < \infty, \quad \forall T > 0 \quad (49)$$

and

$$\int_H \varphi(x) \mu_t(dx) - \int_H \varphi(x) \mu(dx) = \int_0^t \left( \int_H K \varphi(x) \mu_s(dx) \right) ds, \quad (50)$$

$t \geq 0$ ,  $\varphi \in D(K)$ . Finally, the solution of (49), (50) is given by  $\{P_t^* \mu\}_{t \geq 0}$ .

It worth to note that  $C_b(H) \subset C_{b,1}(H) \subset C_{b,d}(L^{2d}(\mathcal{O}))$ , with continuous embedding. This argument will be used in what follows. Note, also, that for any  $\varphi \in C_{b,d}(L^{2d}(\mathcal{O}))$  there exists a sequence  $(\varphi_n)_{n \in \mathbb{N}} \subset C_b(H)$  such that

$$\lim_{n \rightarrow \infty} \frac{\varphi_n}{1 + |\cdot|_{L^{2d}(\mathcal{O})}^d} \stackrel{\pi}{=} \frac{\varphi}{1 + |\cdot|_{L^{2d}(\mathcal{O})}^d}.$$

In the above formula we have to understand  $(1 + |x|_{L^{2d}(\mathcal{O})}^d)^{-1} \varphi(x) = 0$  if  $x \in H \setminus L^{2d}(\mathcal{O})$ . This allow us to use the  $\pi$ -convergence also for functions belonging to the space  $C_{b,d}(L^{2d}(\mathcal{O}))$ . We denote by  $\mathcal{E}_A(H)$  the linear span of the real and imaginary parts of the functions<sup>6</sup>

$$H \rightarrow \mathbb{C}, \quad x \mapsto e^{i\langle x, h \rangle}, \quad h \in D(A).$$

The main result of this section is the following

**Theorem 6.3.** *The infinitesimal operator  $(K, D(K))$  defined in (48) is an extension of  $K_0$ , and for any  $\varphi \in \mathcal{E}_A(H)$  we have  $\varphi \in D(K)$  and  $K\varphi = K_0\varphi$ . Moreover, the set  $\mathcal{E}_A(H)$  is a  $\pi$ -core for  $(K, D(K))$ , that is for any  $\varphi \in D(K)$  there exist  $m \in \mathbb{N}$  and an  $m$ -indexed sequence  $(\varphi_{n_1, \dots, n_m})_{n_1 \in \mathbb{N}, \dots, n_m \in \mathbb{N}} \subset \mathcal{E}_A(H)$  such that*

$$\lim_{n_1 \rightarrow \infty} \cdots \lim_{n_m \rightarrow \infty} \frac{\varphi_{n_1, \dots, n_m}}{1 + |\cdot|_{L^{2d}(\mathcal{O})}^d} \stackrel{\pi}{=} \frac{\varphi}{1 + |\cdot|_{L^{2d}(\mathcal{O})}^d} \quad (51)$$

and

$$\lim_{n_1 \rightarrow \infty} \cdots \lim_{n_m \rightarrow \infty} \frac{K_0 \varphi_{n_1, \dots, n_m}}{1 + |\cdot|_{L^{2d}(\mathcal{O})}^d} \stackrel{\pi}{=} \frac{K\varphi}{1 + |\cdot|_{L^{2d}(\mathcal{O})}^d}. \quad (52)$$

Thanks to Theorem 6.3 we are able to prove the following

---

<sup>6</sup>Here  $A$  is self-adjoint, hence we take  $h \in D(A)$  (cf section 1).

**Theorem 6.4.** *For any  $\mu \in \mathcal{M}_d(L^{2d}(\mathcal{O}))$  there exists an unique family of measures  $\{\mu_t\}_{t \geq 0} \subset \mathcal{M}_d(L^{2d}(\mathcal{O}))$  fulfilling (49) and the measure equation*

$$\int_H \varphi(x) \mu_t(dx) - \int_H \varphi(x) \mu(dx) = \int_0^t \left( \int_H K_0 \varphi(x) \mu_s(dx) \right) ds, \quad (53)$$

*$t \geq 0$ ,  $\varphi \in \mathcal{E}_A(H)$ . Finally, the solution of (49), (53) is given by  $\{P_t^* \mu\}_{t \geq 0}$ .*

In the next section we study the transition semigroup (47) and its infinitesimal generator (48) in the space  $C_{b,d}(L^{2d}(\mathcal{O}))$ . In section 6.3 we shall introduce an approximation of problem (42) that will be often used in what follows. Finally, in sections 6.4, 6.5, 6.6 we prove Theorems 6.2, 6.3, 6.4, respectively.

## 6.2 The transition semigroup in $C_{b,d}(L^{2d}(\mathcal{O}))$

The following two propositions may be proved in much the same way as Proposition 2.2 and Proposition 2.4.

**Proposition 6.5.** *Formula (47) semigroup of operators  $(P_t)_{t \geq 0}$  in  $C_{b,d}(L^{2d}(\mathcal{O}))$ , and there exists a family of probability measures  $\{\pi_t(x, \cdot), t \geq 0, x \in L^{2d}(\mathcal{O})\} \subset \mathcal{M}_d(L^{2d}(\mathcal{O}))$  and two constants  $c_0, \omega_0 > 0$ , such that*

- (i)  $P_t \in \mathcal{L}(C_{b,d}(L^{2d}(\mathcal{O})))$  and  $\|P_t\|_{\mathcal{L}(C_{b,d}(L^{2d}(\mathcal{O})))} \leq c_0 e^{\omega_0 t}$ ;
- (ii)  $P_t \varphi(x) = \int_H \varphi(y) \pi_t(x, dy)$ , for any  $t \geq 0$ ,  $\varphi \in C_{b,d}(L^{2d}(\mathcal{O}))$ ,  $x \in L^{2d}(\mathcal{O})$ ;
- (iii) for any  $\varphi \in C_{b,d}(L^{2d}(\mathcal{O}))$ ,  $x \in H$ , the function  $\mathbb{R}^+ \rightarrow \mathbb{R}$ ,  $t \mapsto P_t \varphi(x)$  is continuous.
- (iv)  $P_t P_s = P_{t+s}$ , for any  $t, s \geq 0$  and  $P_0 = I$ ;
- (v) for any  $\varphi \in C_{b,d}(L^{2d}(\mathcal{O}))$  and any sequence  $(\varphi_n)_{n \in \mathbb{N}} \subset C_{b,d}(L^{2d}(\mathcal{O}))$  such that

$$\lim_{n \rightarrow \infty} \frac{\varphi_n}{1 + |\cdot|_{L^{2d}(\mathcal{O})}^d} \stackrel{\pi}{=} \frac{\varphi}{1 + |\cdot|_{L^{2d}(\mathcal{O})}^d}$$

*we have, for any  $t \geq 0$ ,*

$$\lim_{n \rightarrow \infty} \frac{P_t \varphi_n}{1 + |\cdot|_{L^{2d}(\mathcal{O})}^d} \stackrel{\pi}{=} \frac{P_t \varphi}{1 + |\cdot|_{L^{2d}(\mathcal{O})}^d}.$$

**Proposition 6.6.** *Under the hypothesis of Proposition 6.5, let  $(K, D(K))$  be the infinitesimal generator (48). Then*

- (i) *for any  $\varphi \in D(K)$ , we have  $P_t\varphi \in D(K)$  and  $KP_t\varphi = P_tK\varphi$ ,  $t \geq 0$ ;*
- (ii) *for any  $\varphi \in D(K)$ ,  $x \in L^{2d}(\mathcal{O})$ , the map  $[0, \infty) \rightarrow \mathbb{R}$ ,  $t \mapsto P_t\varphi(x)$  is continuously differentiable and  $(d/dt)P_t\varphi(x) = P_tK\varphi(x)$ ;*
- (iii) *given  $c_0, \omega_0 > 0$  as in Proposition 6.5, for any  $\omega > \omega_0$  the linear operator  $R(\omega, K)$  on  $C_{b,d}(L^{2d}(\mathcal{O}))$  done by*

$$R(\omega, K)f(x) = \int_0^\infty e^{-\omega t} P_t f(x) dt, \quad f \in C_{b,d}(L^{2d}(\mathcal{O})), x \in L^{2d}(\mathcal{O})$$

*satisfies, for any  $f \in C_{b,1}(H)$*

$$R(\omega, K) \in \mathcal{L}(C_{b,d}(L^{2d}(\mathcal{O}))), \quad \|R(\omega, K)\|_{\mathcal{L}(C_{b,d}(L^{2d}(\mathcal{O})))} \leq \frac{c_0}{\omega - \omega_0}$$

$$R(\omega, K)f \in D(K), \quad (\omega I - K)R(\omega, K)f = f.$$

*We call  $R(\omega, K)$  the resolvent of  $K$  at  $\omega$ .*

### 6.3 Some auxiliary results

It is convenient to consider the Ornstein–Uhlenbeck process

$$\begin{cases} dZ(t) = AZ(t)dt + (-A)^{-\gamma/2}dW(t), \\ Z(0) = x, \end{cases}$$

and the corresponding transition semigroup in  $C_{b,1}(H)$

$$R_t\varphi(x) = \mathbb{E}[\varphi(Z(t, x))], \quad \varphi \in C_{b,1}(H). \quad (54)$$

Notice that thanks to (40), (41) the operator

$$\begin{aligned} Q_t x &= \int_0^t e^{sA} B B^* e^{sA^*} x ds = \int_0^t (-A)^{-\gamma} e^{2tA} x dt \\ &= \frac{1}{2}(-A)^{-(1+\gamma)}(1 - e^{2tA})x, \quad t \geq 0, x \in H, \end{aligned}$$

is of trace class. This implies that the Ornstein–Uhlenbeck process  $Z(t, x)$  has gaussian law of mean  $e^{tA}x$  and covariance operator  $Q_t$ , and the representation formula

$$R_t\varphi(x) = \int_H \varphi(e^{tA}x + y) \mathcal{N}_{Q_t}(dy)$$

holds for any  $t \geq 0$ ,  $\varphi \in C_{b,1}(H)$ ,  $x \in H$ . Notice that we can take  $\gamma = 0$  and  $B = I$  (white noise) only for  $n = 1$ . As in section 4.1, we denote by  $(L, D(L))$  the infinitesimal generator of the Ornstein-Uhlenbeck semigroup  $(R_t)_{t \geq 0}$  in the space  $C_{b,1}(H)$ .

A basic tool we use to prove our results is provided by the following approximating problem

$$\begin{cases} dX^n(t) = (AX^n(t) + F_n(X^n(t))dt + (-A)^{-\gamma/2}dW(t), \\ X^n(0) = x \in H, \end{cases} \quad (55)$$

where for any  $n \in \mathbb{N}$ ,  $F_n$  is defined by

$$F_n(x)(\xi) = \lambda x(\xi) - p_n(x(\xi)),$$

and  $p_n$  is defined by

$$p_n(\eta) = \frac{np(\eta)}{\sqrt{n^2 + p^2(\eta)}}, \quad \eta \in \mathbb{R}.$$

Notice that  $p_n$  is bounded and differentiable, with bounded derivative

$$p'_n(\eta) = \frac{np'(\eta)}{\sqrt{n^2 + p^2(\eta)}} \left( 1 - \frac{p^2(\eta)}{n^2 + p^2(\eta)} \right) \geq 0,$$

for any  $n \in \mathbb{N}$ ,  $\eta \in \mathbb{R}$ . Clearly,  $|p_n(\eta)| \leq |p(\eta)|$ ,  $\eta \in \mathbb{R}$  and  $p_n(\eta) \rightarrow p(\eta)$  as  $n \rightarrow \infty$ , for any  $\eta \in \mathbb{R}$ .  $F_n$  is Lipschitz continuous, and for any  $n \in \mathbb{N}$ ,  $x \in H$  problem (55) has a unique mild solution  $(X^n(t, x))_{t \geq 0}$  (cf section 1). Since by the above discussion we have  $|F_n(x)| \leq |F(x)|$ ,  $x \in H$  and  $|F_n(x)| \rightarrow |F(x)|$  as  $n \rightarrow \infty$ , for any  $x \in H$  it is not difficult but tedious to show that for any  $x \in L^{2d}(\mathcal{O})$  it holds

$$\lim_{n \rightarrow \infty} \sup_{t \in [0, T]} \mathbb{E} [|X^n(t, x) - X(t, x)|^2] = 0 \quad (56)$$

and

$$\mathbb{E} [|X^n(t, x)|_{L^{2d}(\mathcal{O})}^d] \leq \mathbb{E} [|X(t, x)|_{L^{2d}(\mathcal{O})}^d], \quad n \in \mathbb{N}. \quad (57)$$

**Proposition 6.7.** *For any  $n \in \mathbb{N}$ , let  $(P_t^n)_{t \geq 0}$  be the transitional semigroup associated to the mild solution of problem (55) in the space  $C_{b,d}(L^{2d}(\mathcal{O}))$ , defined as in (47) with  $X^n(t, x)$  replacing  $X(t, x)$ . Then*

- (i)  $(P_t^n)_{t \geq 0}$  satisfies statements (i)–(v) of Proposition 6.5, and for  $c_0, \omega_0$  as in Proposition 6.5 we have  $\|P_t^n\|_{\mathcal{L}(C_{b,d}(L^{2d}(\mathcal{O})))} \leq c_0 e^{\omega_0 t}$ ;

- (ii)  $(P_t^n)_{t \geq 0}$  is a semigroup of operators in the space  $C_{b,1}(H)$ , and it satisfies statements (i)–(v) of Proposition 2.2. In particular, there exists  $c_n, \omega_n > 0$  such that  $\|P_t^n\|_{\mathcal{L}(C_{b,1}(H))} \leq c_n e^{\omega_n t}$ , for any  $t \geq 0$ .

*Proof.* (i) follows by (57). (ii) follows since equation (55) satisfies Hypothesis 1.1.  $\square$

By (ii) of Proposition 6.7, we can define, for any  $n \in \mathbb{N}$ , the infinitesimal generator  $(K_n, D(K_n))$  of the semigroup  $(P_t^n)_{t \geq 0}$  in the space  $C_{b,1}(H)$  (cf (6)).

By Theorem 4.1 and Proposition 6.7 it follows

**Proposition 6.8.** *For any  $n \in \mathbb{N}$  we have  $D(L) \cap C_b^1(H) = D(K_n) \cap C_b^1(H)$ , and for any  $\varphi \in D(L) \cap C_b^1(H)$  we have  $K_n \varphi = L\varphi + \langle D\varphi, F_n \rangle$ .*

The semigroup  $(P_t^n)_{t \geq 0}$  enjoys the following property, which will be essential in the proof of Theorem 6.3.

**Proposition 6.9.** *For any  $n \in \mathbb{N}$ , the semigroup  $(P_t^n)_{t \geq 0}$  maps  $C_b^1(H)$  into  $C_b^1(H)$ , and for any  $\varphi \in C_b^1(H)$  it holds*

$$|DP_t \varphi(x)| \leq e^{2(\lambda - \pi^2)t} \sup_{x \in H} |D\varphi(x)|$$

*Proof.* Since the nonlinearity  $F_n$  is differentiable with uniformly continuous and bounded differential, it is well known (see, for instance, [7]) that the mild solution  $X^n(t, x)$  of problem (55) is differentiable with respect to  $x$  and for any  $x, h \in H$  we have  $DX^n(t, x) \cdot h = \eta_n^h(t, x)$ , where  $\eta_n^h(t, x)$  is the mild solution of the differential equation with random coefficients

$$\begin{cases} \frac{d}{dt} \eta_n^h(t, x) = A \eta_n^h(t, x) + DF_n(X^n(t, x)) \cdot \eta_n^h(t, x) & t \geq 0 \\ \eta_n^h(t, x) = 0. \end{cases}$$

By multiplying by  $\eta_n^h(t, x)$  and by integrating over  $\mathcal{O}$  we find

$$\frac{1}{2} \frac{d}{dt} |\eta_n^h(t, x)|^2 = \langle (A + \lambda) \eta_n^h(t, x), \eta_n^h(t, x) \rangle - \int_{\mathcal{O}} p'_n(X^n(t, x)(\xi)) |\eta_n^h(t, x)(\xi)|^2 d\xi.$$

By taking into account that  $p'_n \geq 0$  and by integrating by parts we find

$$\frac{1}{2} \frac{d}{dt} |\eta_n^h(t, x)|^2 + \int_{\mathcal{O}} |D_\xi \eta_n^h(t, x)(\xi)|^2 d\xi \leq \lambda |\eta_n^h(t, x)|^2.$$

Now, the classical Poincaré inequality implies  $|D_\xi \eta_n^h(t, x)| \geq \pi^2 |\eta_n^h(t, x)|$  and we obtain

$$\frac{1}{2} \frac{d}{dt} |\eta_n^h(t, x)|^2 \leq (\lambda - \pi^2) |\eta_n^h(t, x)|^2, \quad x \in H, t \geq 0.$$

Consequently, by the Gronwall lemma we find

$$|\eta^h(t, x)| \leq e^{2(\lambda - \pi^2)t} |h|. \quad (58)$$

Now take  $\varphi \in C_b^1(H)$ . For any  $x, h \in H$  we have

$$DP_t^n \varphi(x) \cdot h = \mathbb{E} [D\varphi(X^n(t, x)) \cdot \eta^h(t, x)].$$

Hence by (58)

$$|DP_t^n \varphi(x) \cdot h| \leq \mathbb{E} [|D\varphi(X^n(t, x))| |\eta^h(t, x)|] \leq \sup_{x \in H} |D\varphi(x)| e^{2(\lambda - \pi^2)t} |h|,$$

which implies the result.  $\square$

## 6.4 Proof of Theorem 6.2

We have first to show that  $(P_t^*)_{t \geq 0}$  is a semigroup of linear and continuous operators in  $(C_{b,d}(L^{2d}(\mathcal{O})))^*$  and that  $P_t^* \mu \in \mathcal{M}_d(L^{2d}(\mathcal{O}))$  for any  $t \geq 0$ ,  $\mu \in \mathcal{M}_d(L^{2d}(\mathcal{O}))$ . These facts follow by Proposition 6.5 and by the argument of Lemma 3.2. We left the details to the reader.

We now show existence of a solution for the measure equation, namely we show that  $\{P_t^* \mu\}_{t \geq 0}$  fulfils (53), (49). To show that  $\{P_t^* \mu\}_{t \geq 0}$  fulfils (53) it can be used the argument in Lemma 3.3. We left the details to the reader. We now check that (49) holds. Fix  $T > 0$ . By the local boundedness of the operators  $P_t^* \mu$  and by the semigroup property it follows that there exists  $c > 0$  such that

$$\sup_{t \in [0, T]} \|P_t^*\|_{\mathcal{L}((C_{b,d}(L^{2d}(\mathcal{O})))^*)} \leq c.$$

Still by the first part of the theorem, since  $\mu \in \mathcal{M}_d(L^{2d}(\mathcal{O}))$  we have  $P_t^* \mu \in \mathcal{M}_d(L^{2d}(\mathcal{O}))$ . Hence

$$\begin{aligned} \int_0^T \left( \int_H |x|_{L^{2d}(\mathcal{O})}^d |P_t^* \mu|(dx) \right) dt &= \int_0^T \left( \int_{L^{2d}(\mathcal{O})} |x|_{L^{2d}(\mathcal{O})}^d |P_t^* \mu|(dx) \right) dt \\ &\leq \int_0^T \|P_t^* \mu\|_{(C_{b,d}(L^{2d}(\mathcal{O})))^*} dt \leq c \int_0^T \|\mu\|_{(C_{b,d}(L^{2d}(\mathcal{O})))^*} dt \\ &= cT \|\mu\|_{(C_{b,d}(L^{2d}(\mathcal{O})))^*} = cT \int_{L^{2d}(\mathcal{O})} (1 + |x|_{L^{2d}(\mathcal{O})}^d) |\mu|(dx) < \infty. \end{aligned}$$

Then, (49) is proved.

We now prove uniqueness of the solution. By (3) follows that the mild solution  $X(t, x)$  of problem (43) can be extended to a process  $(X(t, x))_{t \geq 0, x \in H}$  with values in  $H$  and adapted to the filtration  $(\mathcal{F}_t)_{t \geq 0}$ . In the literature, the process  $X(t, x)$  is called a *generalized solution* of equation (43) (see [6]). Hence, we can extend the transition semigroup (47) to a semigroup in  $C_b(H)$ , still denoted by  $(P_t)_{t \geq 0}$ , by setting

$$P_t \varphi(x) = \mathbb{E}[\varphi(X(t, x))] \quad t \geq 0, x \in H, \varphi \in C_b(H).$$

Clearly,  $\|P_t\|_{\mathcal{L}(C_b(H))} \leq 1$ . In addition, the representation

$$P_t \varphi(x) = \int_H \varphi(y) \pi'_t(x, dy)$$

holds for any  $\varphi \in C_b(H)$ , where  $\pi'_t(x, \cdot)$  is the probability measure on  $H$  defined by  $\pi'_t(x, \Gamma) = \mathbb{P}(X(t, x) \in \Gamma)$ ,  $\Gamma \in \mathcal{B}(H)$ . It is clear that  $\pi'_t(x, \Gamma) = \pi_t(x, \Gamma)$  when  $\Gamma \in \mathcal{B}(L^{2d}(\mathcal{O}))$ . We define the infinitesimal generator  $K : D(K, C_b(H)) \rightarrow C_b(H)$  of the semigroup  $(P_t)_{t \geq 0}$  in the space  $C_b(H)$  as in (13). By arguing as in Lemma 3.3, the semigroup  $(P_t)_{t \geq 0}$  in  $C_b(H)$  is a stochastically continuous Markov semigroup, in the sense of [9]. So, we can apply Theorem 3.1 and then for any  $\mu \in \mathcal{M}(H)$  there exists a unique family of measures  $\{\mu_t\}_{t \geq 0} \subset \mathcal{M}(H)$  such that

$$\int_0^T |\mu_t|(H) dt < \infty, \quad \forall T > 0 \quad (59)$$

and (50) hold for any  $t \geq 0$ ,  $\varphi \in D(K, C_b(H))$ .

Now take  $\mu = 0$ , and assume that  $\{\mu_t\}_{t \geq 0} \subset \mathcal{M}_d(L^{2d}(\mathcal{O}))$  fulfils (53), (49). Since  $\{\mu_t\}_{t \geq 0} \subset \mathcal{M}(H)$ , we want to show that  $\{\mu_t\}_{t \geq 0}$  fulfils also (59) and (50) for any  $t \geq 0$ ,  $\varphi \in D(K, C_b(H))$ . Taking in mind that for this equation the solution is unique, this will imply  $\mu_t = 0$  (as measure in  $H$  and consequently as measure in  $L^{2d}(\mathcal{O})$ ) for any  $t \geq 0$ .

Clearly, (59) follows by (53). It is also possible to prove, by a standard argument, that  $D(K, C_b(H)) \subset D(K)$  and  $D(K, C_b(H)) = \{\varphi \in D(K) : K\varphi \in C_b(H)\}$ . Then, for any  $\varphi \in D(K, C_b(H))$ , we have  $\varphi \in D(K)$  and hence (50) holds for any  $\varphi \in D(K, C_b(H))$ . This concludes the proof.  $\square$

## 6.5 Proof of Theorem 6.3

The proof is splitted into two lemmata.



**Lemma 6.10.** *Let  $(K, D(K))$  be the infinitesimal generator (48). We have  $D(L) \cap C_b^1(H) \subset D(K) \cap C_b^1(H)$  and  $K\varphi(x) = L\varphi(x) + \langle D\varphi(x), F(x) \rangle$  for any  $\varphi \in D(L) \cap C_b^1(H)$ ,  $x \in L^{2d}(\mathcal{O})$ . Moreover,  $(K, D(K))$  is an extension of  $K_0$ , and for any  $\varphi \in \mathcal{E}_A(H)$  we have  $\varphi \in D(K)$  and  $K\varphi = K_0\varphi$ .*

*Proof.* Take  $\varphi \in \mathcal{E}_A(H)$ . We recall that  $\mathcal{E}_A(H) \subset C_b^1(H) \cap D(L)$ , where  $(L, D(L))$  was introduced in section 6.3. We also stress that since  $L^{2d}(\mathcal{O}) \subset H$ , then  $D(L) \subset C_{b,1}(H) \subset C_{b,d}(L^{2d}(\mathcal{O}))$  with continuous embedding. This allow us to proceed as in Theorem 4.1 to find

$$R_t\varphi(x) - \varphi(x) = P_t\varphi(x) - \varphi(x)$$

$$-\mathbb{E} \left[ \int_0^1 \left\langle D\varphi(\xi Z(t, x) + (1 - \xi)X(t, x)), \int_0^t e^{(t-s)A} F(X(s, x)) ds \right\rangle d\xi \right],$$

for any  $x \in L^{2d}(\mathcal{O})$ . Hence, by taking into account that  $\varphi \in D(L)$ , it follows easily that for any  $x \in L^{2d}(\mathcal{O})$

$$\lim_{t \rightarrow 0^+} \frac{P_t\varphi(x) - \varphi(x)}{t} = L\varphi(x) + \langle D\varphi(x), F(x) \rangle.$$

Since there exists  $c > 0$  such that  $|F(x)| \leq c|x|_{L^{2d}(\mathcal{O})}^d$ ,  $x \in L^{2d}(\mathcal{O})$ , it follows

$$\begin{aligned} & \sup_{t \in (0,1]} \left\| \frac{P_t\varphi - \varphi}{t} \right\|_{C_{b,d}(L^{2d}(\mathcal{O}))} \\ & \leq \sup_{t \in (0,1]} \left\| \frac{R_t\varphi - \varphi}{t} \right\|_{0,1} + \sup_{x \in H} \|D\varphi(x)\|_{\mathcal{L}(H)} \sup_{x \in L^{2d}(\mathcal{O})} \frac{|F(x)|}{1 + |x|_{L^{2d}(\mathcal{O})}^d} < \infty, \end{aligned}$$

that implies  $\varphi \in D(K)$ . This proves the first statement. The fact that  $(K, D(K))$  is an extension of  $K_0$  follows by Proposition 4.3.  $\square$

**Lemma 6.11.** *The set  $\mathcal{E}_A(H)$  is a  $\pi$ -core for  $(K, D(K))$ , and for any  $\varphi \in D(K)$  there exists  $m \in \mathbb{N}$  and an  $m$ -indexed sequence  $(\varphi_{n_1, \dots, n_m}) \subset \mathcal{E}_A(H)$  such that*

$$\lim_{n_1 \rightarrow \infty} \cdots \lim_{n_m \rightarrow \infty} \frac{\varphi_{n_1, \dots, n_m}}{1 + |\cdot|_{L^{2d}(\mathcal{O})}^d} \stackrel{\pi}{=} \frac{\varphi}{1 + |\cdot|_{L^{2d}(\mathcal{O})}^d}, \quad (60)$$

$$\lim_{n_1 \rightarrow \infty} \cdots \lim_{n_m \rightarrow \infty} \frac{K_0\varphi_{n_1, \dots, n_m}}{1 + |\cdot|_{L^{2d}(\mathcal{O})}^d} \stackrel{\pi}{=} \frac{K\varphi}{1 + |\cdot|_{L^{2d}(\mathcal{O})}^d}. \quad (61)$$

*Proof.* Take  $\varphi \in D(K)$ . We shall construct the claimed sequence in four steps.

**Step 1.** Fix  $\omega > \omega_0, 2(\lambda - \pi^2)$  and set  $f = \omega\varphi - K\varphi$ . Then we have  $\varphi = R(\omega, K)f$ . We approximate  $f$  as follows: for any  $n_1 \in \mathbb{N}$  we set

$$f_{n_1}(x) = \frac{n_1 f(e^{\frac{1}{n_1}A}x)}{n_1 + |e^{\frac{1}{n_1}A}x|_{L^{2d}(\mathcal{O})}^d}, \quad x \in H$$

By the well known properties of the heat semigroup, we have  $e^{\frac{1}{n_1}A}x \in L^{2d}(\mathcal{O})$ , for any  $x \in H$ . Hence,  $f_{n_1} \in C_b(H)$  and

$$\lim_{n_1 \rightarrow \infty} \frac{f_{n_1}}{1 + |\cdot|_{L^{2d}(\mathcal{O})}^d} \stackrel{\pi}{=} \frac{f}{1 + |\cdot|_{L^{2d}(\mathcal{O})}^d}.$$

By Proposition 6.5 we have

$$\lim_{n_1 \rightarrow \infty} \frac{P_t f_{n_1}}{1 + |\cdot|_{L^{2d}(\mathcal{O})}^d} \stackrel{\pi}{=} \frac{P_t f}{1 + |\cdot|_{L^{2d}(\mathcal{O})}^d}$$

for any  $t \geq 0$ . Since we have  $\|P_t\|_{\mathcal{L}(C_{b,d}(L^{2d}(\mathcal{O})))} \leq c_0 e^{\omega_0 t}$ ,  $\forall t \geq 0$  (cf (i) of Proposition 6.5) and  $\omega > \omega_0$ , it follows

$$\lim_{n_1 \rightarrow \infty} \frac{R(\omega, K)f_{n_1}}{1 + |\cdot|_{L^{2d}(\mathcal{O})}^d} \stackrel{\pi}{=} \frac{R(\omega, K)f}{1 + |\cdot|_{L^{2d}(\mathcal{O})}^d}.$$

Setting  $\varphi_{n_1} = R(\omega, K)f_{n_1}$ , by the above argument we have

$$\lim_{n_1 \rightarrow \infty} \frac{\varphi_{n_1}}{1 + |\cdot|_{L^{2d}(\mathcal{O})}^d} \stackrel{\pi}{=} \frac{\varphi}{1 + |\cdot|_{L^{2d}(\mathcal{O})}^d}, \quad \lim_{n_1 \rightarrow \infty} \frac{K\varphi_{n_1}}{1 + |\cdot|_{L^{2d}(\mathcal{O})}^d} \stackrel{\pi}{=} \frac{K\varphi}{1 + |\cdot|_{L^{2d}(\mathcal{O})}^d}. \quad (62)$$

**Step 2.** For any  $n_1 \in \mathbb{N}$ , let us fix a sequence  $(f_{n_1, n_2})_{n_2 \in \mathbb{N}} \subset C_b^1(H)$  such that

$$\lim_{n_2 \rightarrow \infty} f_{n_1, n_2} \stackrel{\pi}{=} f_{n_1}.$$

Now set  $\varphi_{n_1, n_2} = R(\omega, K)f_{n_1, n_2}$ . By arguing as in step 1 we have

$$\lim_{n_2 \rightarrow \infty} \varphi_{n_1, n_2} \stackrel{\pi}{=} \varphi_{n_1}, \quad \lim_{n_2 \rightarrow \infty} K\varphi_{n_1, n_2} \stackrel{\pi}{=} K\varphi_{n_1}. \quad (63)$$

**Step 3.** We now consider the approximation of  $K$ . We denote by  $(K_{n_3}, D(K_{n_3}))$  the infinitesimal generator of the transition semigroup associated to the mild solution of problem (55) in the space  $C_{b,1}(H)$ . For any  $n_1, n_2, n_3 \in \mathbb{N}$  set

$$\varphi_{n_1, n_2, n_3} = \int_0^\infty e^{-\omega t} P_t^{n_3} f_{n_1, n_2} dt.$$

Note that in the right-hand side we have not the *resolvent* operator of  $K_{n_3}$  in  $C_{b,1}(H)$  (cf Proposition 2.4, 6.7). For any  $n_1, n_2, n_3 \in \mathbb{N}$  the function  $\varphi_{n_1, n_2, n_3}$  is bounded, since

$$\left| \int_0^\infty e^{-\omega t} P_t^{n_3} f_{n_1, n_2} dt \right| \leq \|f\|_0 \int_0^\infty e^{-\omega t} dt < \infty.$$

The fact that  $\varphi_{n_1, n_2, n_3} \in C_b(H)$  follows by standard computations. By (v) of Proposition 2.2 and by (i) of Proposition 6.7 it follows

$$\lim_{n_3 \rightarrow \infty} \frac{\varphi_{n_1, n_2, n_3}}{1 + |\cdot|_{L^{2d}(\mathcal{O})}^d} \stackrel{\pi}{=} \frac{\varphi_{n_1, n_2}}{1 + |\cdot|_{L^{2d}(\mathcal{O})}^d}, \quad (64)$$

It is also standard to show that  $\varphi_{n_1, n_2, n_3} \in D(K_{n_3})$  and  $K_{n_3} \varphi_{n_1, n_2, n_3} = \omega \varphi_{n_1, n_2, n_3} - f_{n_1, n_2}$ . Hence, by (64) we obtain

$$\lim_{n_3 \rightarrow \infty} \frac{K_{n_3} \varphi_{n_1, n_2, n_3}}{1 + |\cdot|_{L^{2d}(\mathcal{O})}^d} \stackrel{\pi}{=} \frac{K \varphi_{n_1, n_2}}{1 + |\cdot|_{L^{2d}(\mathcal{O})}^d} \quad (65)$$

By Proposition 6.9 it follows that  $\varphi_{n_1, n_2, n_3} \in C_b^1(H)$  and

$$\begin{aligned} |D\varphi_{n_1, n_2, n_3}(x)| &= \left| \int_0^\infty e^{-\omega t} D P_t^{n_3} f_{n_1, n_2}(x) dt \right| \\ &\leq \int_0^\infty e^{-(\omega - 2\lambda + 2\pi^2)t} dt \sup_{x \in H} |Df_{n_1, n_2}(x)| \leq \frac{\sup_{x \in H} |Df_{n_1, n_2}(x)|}{\omega - 2(\lambda - \pi^2)}. \end{aligned} \quad (66)$$

Hence  $\varphi_{n_1, n_2, n_3} \in D(K_{n_3}) \cap C_b^1(H)$ , and by Proposition 6.8 it follows that  $K_{n_3} \varphi_{n_1, n_2, n_3} = L\varphi_{n_1, n_2, n_3} + \langle D\varphi_{n_1, n_2, n_3}, F_{n_3} \rangle$ . Hence, by Lemma 6.10 we have, for any  $x \in L^{2d}(\mathcal{O})$

$$\begin{aligned} K\varphi_{n_1, n_2, n_3}(x) &= L\varphi_{n_1, n_2, n_3}(x) + \langle D\varphi_{n_1, n_2, n_3}(x), F(x) \rangle \\ &= K_{n_3} \varphi_{n_1, n_2, n_3}(x) + \langle D\varphi_{n_1, n_2, n_3}(x), F(x) - F_{n_3}(x) \rangle. \end{aligned} \quad (67)$$

We recall that  $|F_{n_3}(x)| \leq |F(x)| \leq c|x|_{L^{2d}(\mathcal{O})}^d$ , for any  $n_3 \in \mathbb{N}$ ,  $x \in L^{2d}(\mathcal{O})$  and for some  $c > 0$ . In addition,  $|F_{n_3}(x) - F(x)| \rightarrow 0$  as  $n_3 \rightarrow \infty$ , for any  $x \in L^{2d}(\mathcal{O})$ . Consequently, by (66) it follows

$$\lim_{n_3 \rightarrow \infty} \frac{\langle D\varphi_{n_1, n_2, n_3}, F - F_{n_3} \rangle}{1 + |\cdot|_{L^{2d}(\mathcal{O})}^d} \stackrel{\pi}{=} 0. \quad (68)$$

**Step 4.** By Proposition 4.3 for any  $n_1, n_2, n_3 \in \mathbb{N}$  there exists a sequence<sup>7</sup>  $(\varphi_{n_1, n_2, n_3, n_4}) \subset \mathcal{E}_A(H)$  such that

$$\lim_{n_4 \rightarrow \infty} \varphi_{n_1, n_2, n_3, n_4} \stackrel{\pi}{=} \varphi_{n_1, n_2, n_3}, \quad (69)$$

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<sup>7</sup>we assume that it has one index

$$\lim_{n_4 \rightarrow \infty} \frac{\frac{1}{2} \text{Tr}[BB^* D^2 \varphi_{n_1, n_2, n_3, n_4}] + \langle x, A^* D \varphi_{n_1, n_2, n_3, n_4} \rangle}{1 + |\cdot|} \stackrel{\pi}{=} \frac{L \varphi_{n_1, n_2, n_3}}{1 + |\cdot|} \quad (70)$$

and for any  $h \in H$

$$\lim_{n_4 \rightarrow \infty} \langle D \varphi_{n_1, n_2, n_3, n_4}, h \rangle \stackrel{\pi}{=} \langle D \varphi_{n_1, n_2, n_3}, h \rangle.$$

This, together with the above approximation, implies that for any  $n_1, n_2, n_3 \in \mathbb{N}$  we have

$$\lim_{n_4 \rightarrow \infty} \frac{\langle D \varphi_{n_1, n_2, n_3, n_4}, F - F_{n_3} \rangle}{1 + |\cdot|_{L^{2d}(\mathcal{O})}^d} \stackrel{\pi}{=} \frac{\langle D \varphi_{n_1, n_2, n_3}, F - F_{n_3} \rangle}{1 + |\cdot|_{L^{2d}(\mathcal{O})}^d}. \quad (71)$$

**Step 5.** By (62), (63), (64), (69) we have

$$\lim_{n_1 \rightarrow \infty} \lim_{n_2 \rightarrow \infty} \lim_{n_3 \rightarrow \infty} \lim_{n_4 \rightarrow \infty} \frac{\varphi_{n_1, n_2, n_3, n_4}}{1 + |\cdot|_{L^{2d}(\mathcal{O})}^d} \stackrel{\pi}{=} \frac{\varphi}{1 + |\cdot|_{L^{2d}(\mathcal{O})}^d},$$

and consequently (60) follows. We now check

$$\lim_{n_1 \rightarrow \infty} \lim_{n_2 \rightarrow \infty} \lim_{n_3 \rightarrow \infty} \lim_{n_4 \rightarrow \infty} \frac{K_0 \varphi_{n_1, n_2, n_3, n_4}}{1 + |\cdot|_{L^{2d}(\mathcal{O})}^d} \stackrel{\pi}{=} \frac{K \varphi}{1 + |\cdot|_{L^{2d}(\mathcal{O})}^d}.$$

This will prove (61). By Lemma 6.10, for any  $n_1, n_2, n_3, n_4 \in \mathbb{N}$  we have  $K \varphi_{n_1, n_2, n_3, n_4} = K_0 \varphi_{n_1, n_2, n_3, n_4}$ . Moreover, by Theorem 1.3 we have  $\varphi_{n_1, n_2, n_3, n_4} \in D(K_3)$  and by (67)

$$K_0 \varphi_{n_1, n_2, n_3, n_4}(x) = K_{n_3} \varphi_{n_1, n_2, n_3, n_4}(x) + \langle D \varphi_{n_1, n_2, n_3, n_4}(x), F(x) - F_{n_3}(x) \rangle,$$

for any  $n_1, n_2, n_3, n_4 \in \mathbb{N}$ ,  $x \in L^{2d}(\mathcal{O})$ . By (67), (70), (71) it holds

$$\lim_{n_4 \rightarrow \infty} \frac{K_0 \varphi_{n_1, n_2, n_3, n_4}}{1 + |\cdot|_{L^{2d}(\mathcal{O})}^d} \stackrel{\pi}{=} \frac{K_{n_3} \varphi_{n_1, n_2, n_3} + \langle D \varphi_{n_1, n_2, n_3}, F - F_{n_3} \rangle}{1 + |\cdot|_{L^{2d}(\mathcal{O})}^d}.$$

By (65), (68) it holds

$$\lim_{n_3 \rightarrow \infty} \frac{K_{n_3} \varphi_{n_1, n_2, n_3} + \langle D \varphi_{n_1, n_2, n_3}, F - F_{n_3} \rangle}{1 + |\cdot|_{L^{2d}(\mathcal{O})}^d} \stackrel{\pi}{=} \frac{K \varphi_{n_1, n_2}}{1 + |\cdot|_{L^{2d}(\mathcal{O})}^d}.$$

By (62), (63) it holds

$$\lim_{n_1 \rightarrow \infty} \lim_{n_2 \rightarrow \infty} \frac{K \varphi_{n_1, n_2}}{1 + |\cdot|_{L^{2d}(\mathcal{O})}^d} \stackrel{\pi}{=} \frac{K \varphi}{1 + |\cdot|_{L^{2d}(\mathcal{O})}^d}.$$

□

## 6.6 Proof of Theorem 6.4

Take  $\mu \in \mathcal{M}_d(L^{2d}(\mathcal{O}))$ . The fact that  $\{P_t^*\mu\}_{t \geq 0}$  fulfils (49) and (53) follows by Theorems 6.2, 6.3 and by the fact that  $KP_t\varphi = P_tK\varphi = P_tK_0\varphi$ , for any  $\varphi \in \mathcal{E}_A(H)$  (cf Proposition 6.6 and Lemma 6.10). Hence, existence of a solution is proved. Now we show uniqueness. Assume that  $\{\mu_t\}_{t \geq 0} \subset \mathcal{M}_d(L^{2d}(\mathcal{O}))$  fulfils (49) and (53). By Theorem 6.3 for any  $\varphi \in D(K)$  there exist  $m \in \mathbb{N}$  and an  $m$ -indexed sequence  $(\varphi_{n_1, \dots, n_m})_{n_1 \in \mathbb{N}, \dots, n_m \in \mathbb{N}} \subset \mathcal{E}_A(H)$  such that (51), (52) hold. This, together with (49), implies that  $\{\mu_t\}_{t \geq 0}$  fulfils (50) for any  $t \geq 0$ ,  $\varphi \in D(K)$  (here we can use the same argument used to prove Theorem 1.4). Since the solution of (49), (50) is unique and it is given by  $\{P_t^*\mu\}_{t \geq 0}$ , it follows  $\int_H \varphi(x) P_t^*\mu(dx) = \int_H \varphi(x) \mu_t(dx)$ , for any  $\varphi \in \mathcal{E}_A(H)$ . Hence, since  $\mathcal{E}_A(H)$  is  $\pi$ -dense in  $C_b(H)$ , it follows  $\int_H \varphi(x) P_t^*\mu(dx) = \int_H \varphi(x) \mu_t(dx)$ , for any  $\varphi \in C_b(H)$ , that implies  $P_t^*\mu = \mu_t$ ,  $\forall t \geq 0$ . This concludes the proof.  $\square$

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